Signals and Systems

Fourier Series Representation of Periodic Signals

Chang-Su Kim

Introduction

Why do We Need Fourier Analysis?

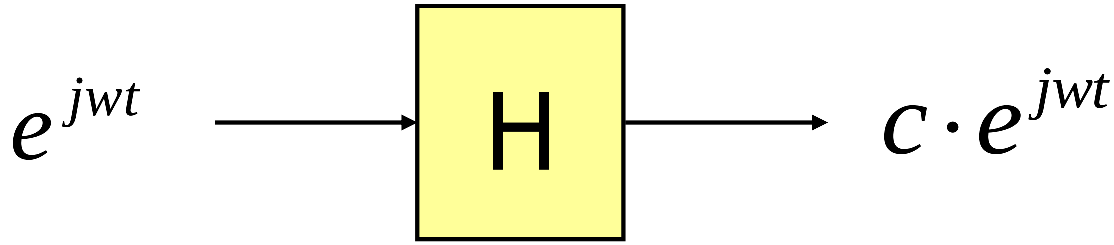
- The essence of Fourier analysis is to represent a signal in terms of complex exponentials

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \ldots + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots$$

- Many reasons:
  - Almost any signal can be represented as a series (or sum or integral) of complex exponentials
    - Signal is periodic
      - Fourier Series (DT, CT)
    - Signal is non-periodic
      - Fourier Transform (DT, CT)

    We will learn these (CTFS, DTFS, CTFT, DTFT) one by one
  - Response of an LTI system to a complex exponential is also a complex exponential with a scaled magnitude.

Response of LTI systems to Complex Exponentials (CT)



$$c = H(jw) = \int_{-\infty}^{\infty} h(\tau) e^{-jw\tau} d\tau$$

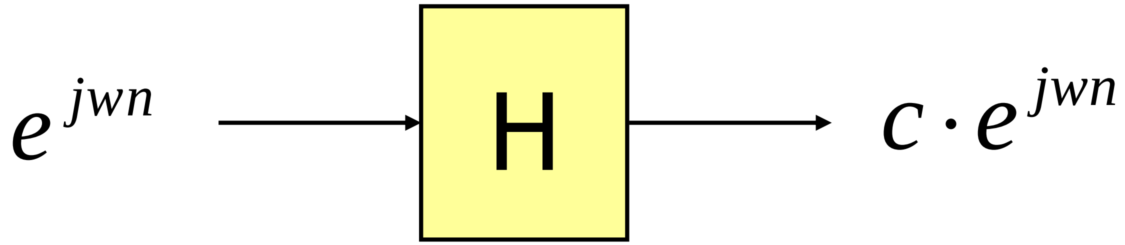
- Property

$$x(t) = a_1 e^{jw_1 t} + a_2 e^{jw_2 t} + a_3 e^{jw_3 t}$$

$$\Rightarrow y(t) = a_1 H(jw_1) e^{jw_1 t} + a_2 H(jw_2) e^{jw_2 t} + a_3 H(jw_3) e^{jw_3 t}$$

  - This is useful because almost every signal can be represented as a sum of complex exponential functions
  - So we can easily compute the response of the system to almost every input signal using the above equation

Response of LTI systems to Complex Exponentials (DT)



$$c = H(e^{jw}) = \sum_{k=-\infty}^{\infty} h[k] e^{-jwk}$$

- Property

$$x[n] = a_1 e^{jw_1 n} + a_2 e^{jw_2 n} + a_3 e^{jw_3 n}$$

$$\Rightarrow y(t) = a_1 H(e^{jw_1}) e^{jw_1 n} + a_2 H(e^{jw_2}) e^{jw_2 n} + a_3 H(e^{jw_3}) e^{jw_3 n}$$

  - This is useful because almost every signal can be represented as a sum of complex exponential functions
  - So we can easily compute the response of the system to almost every input signal using the above equation

Continuous Time Fourier Series

How to represent a periodic function x(t) as

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \ldots + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \ldots$$

	continuous time	discrete time
periodic (series)	**CTFS**	DTFS
aperiodic (transform)	CTFT	DTFT

Harmonically Related Complex Exponentials

- Basic periodic signal

$$e^{j\omega_0 t}$$

  - Fundamental frequency: ω_0
  - Fundamental period: $T = 2\pi / \omega_0$

- The set of harmonically related complex exponentials

$$\{\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t} \mid k = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

- Each $\phi_k(t)$ of the signals is periodic with T

- Thus, a linear combination of them is also periodic with period T:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

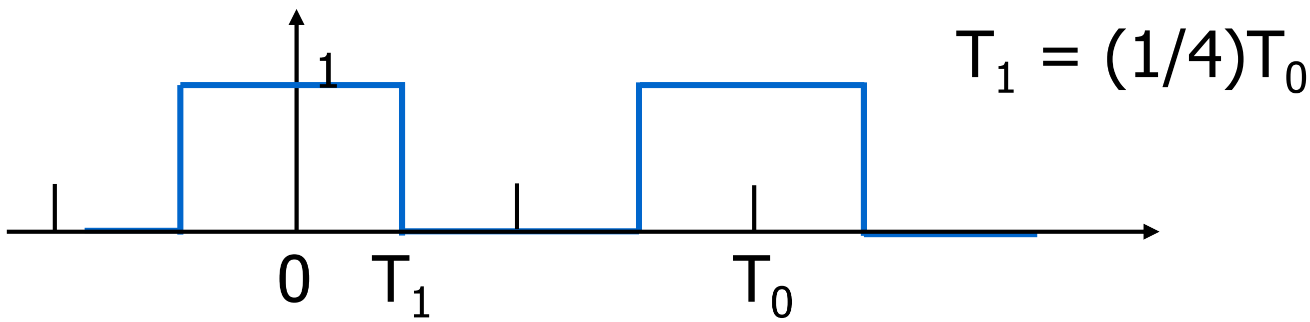
Continuous Time Fourier Series (CTFS)

- Most (all in engineering sense) functions with period $T=2\pi/w_0$ can be represented as a CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Example 1



$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T_0}\int_{-T_1}^{+T_1} e^{-jk\omega_0 t}dt$$

$$= \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{\sin(\frac{k\pi}{2})}{k\pi}.$$

k	...	-5	-4	-3	-2	-1	0	1	2	3	4	5
a_k	...	$1/5\pi$	0	$-1/3\pi$	0	$1/\pi$	1/2	$1/\pi$	0	$-1/3\pi$	0	$1/5\pi$

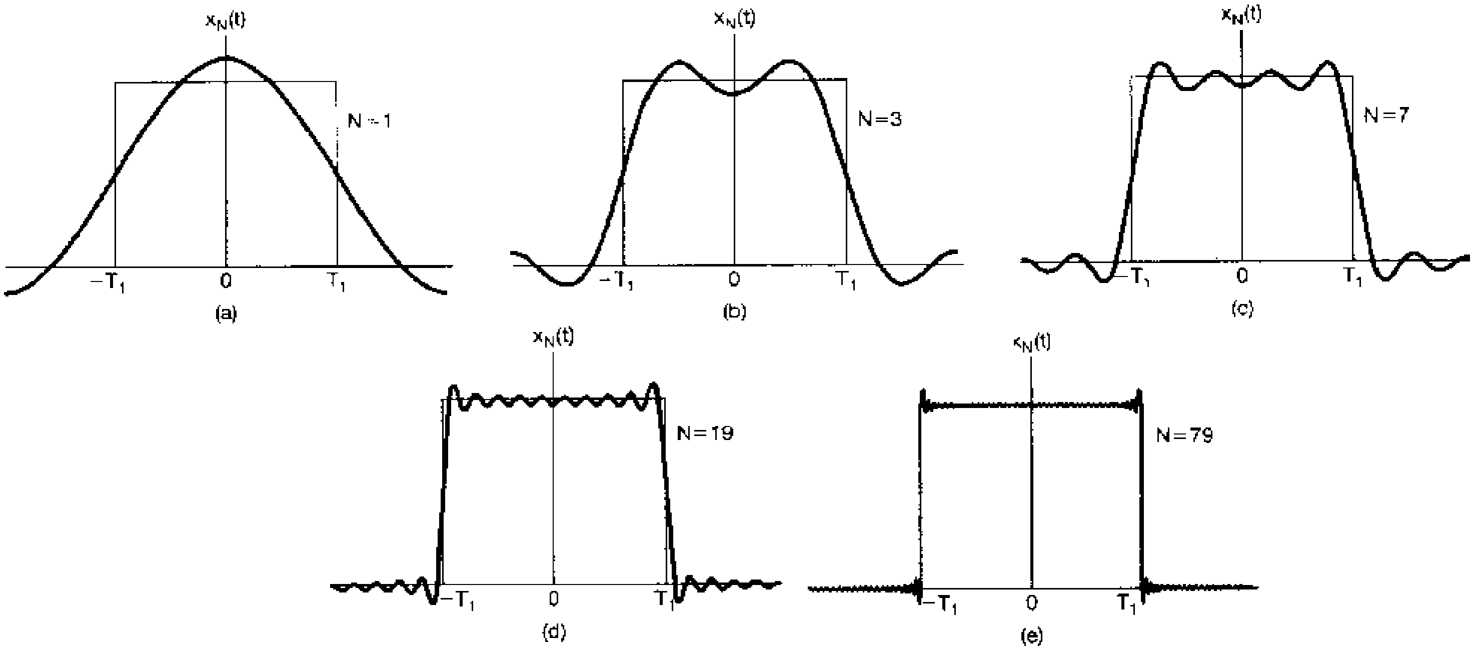
$$x(t) = \cdots - \frac{1}{3\pi}e^{-j3w_0 t} + \frac{1}{\pi}e^{-jw_0 t} + \frac{1}{2} + \frac{1}{\pi}e^{jw_0 t} - \frac{1}{3\pi}e^{j3w_0 t} + \ldots$$

$$= \frac{1}{2} + \frac{2}{\pi}\cos\omega_0 t - \frac{2}{3\pi}\cos 3\omega_0 t + \cdots$$

Example 1 (continued)

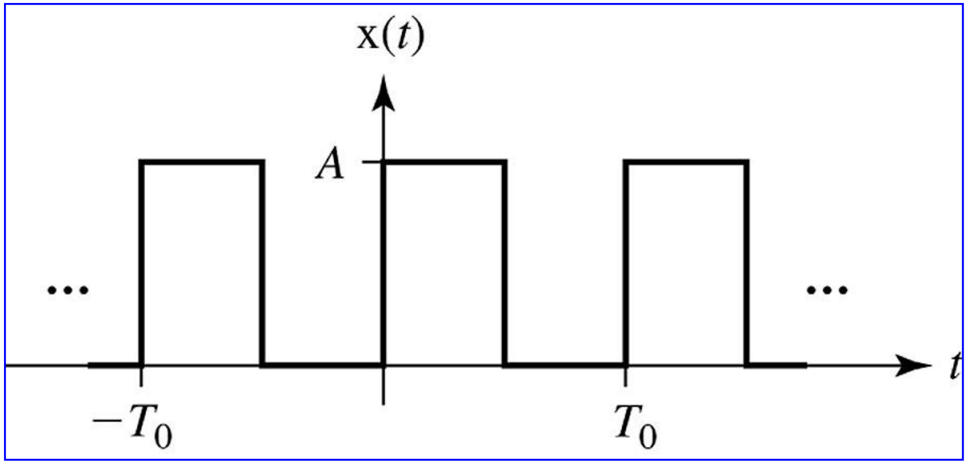
Finite Approximation $x_N(t)$ and Gibbs phenomenon

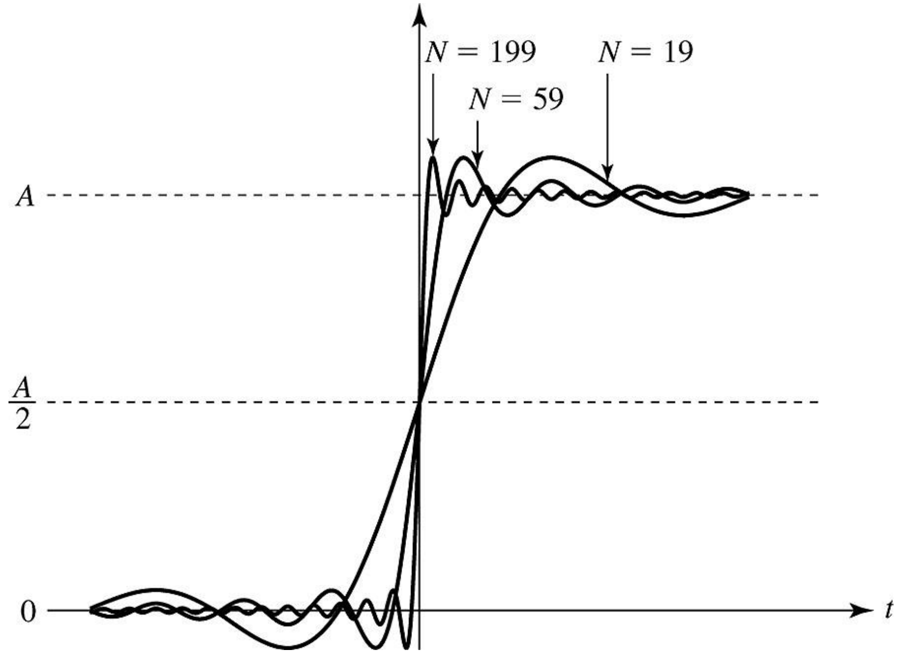
$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

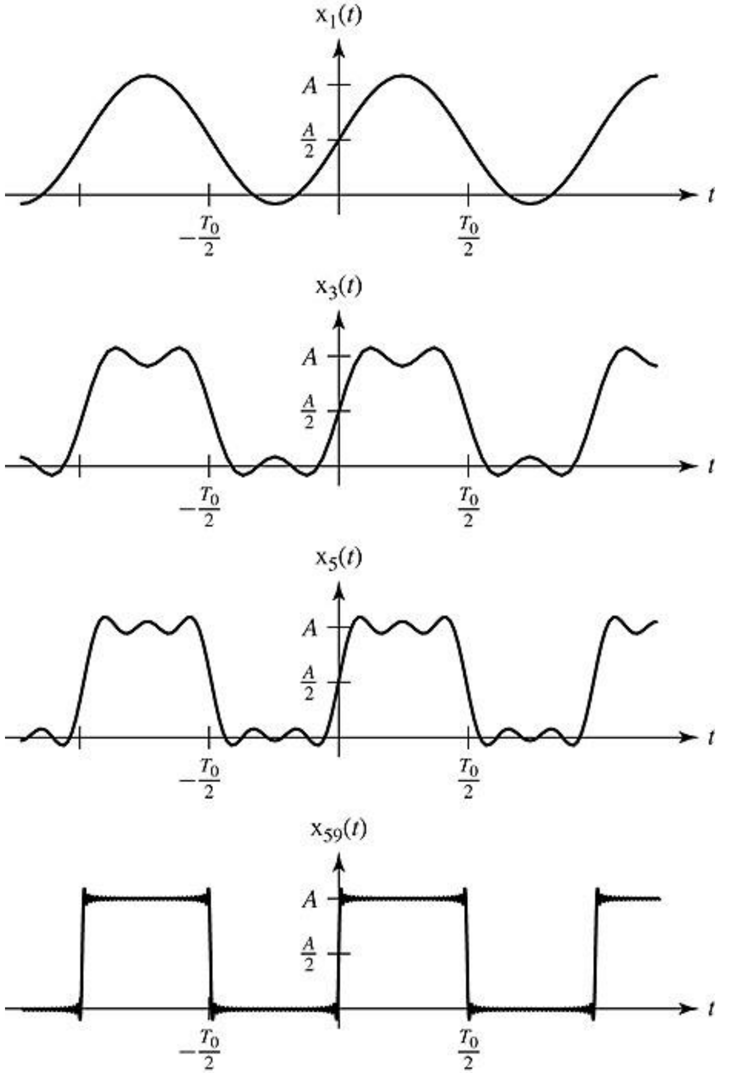


Overshoot is always about 9% of the height of the discontinuity

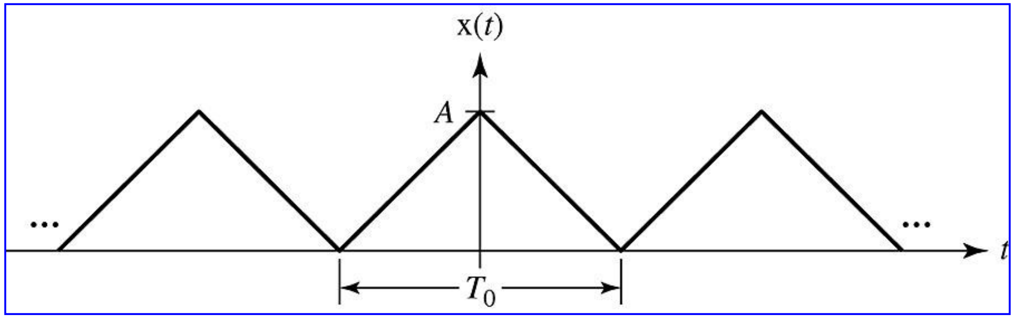
Example 2

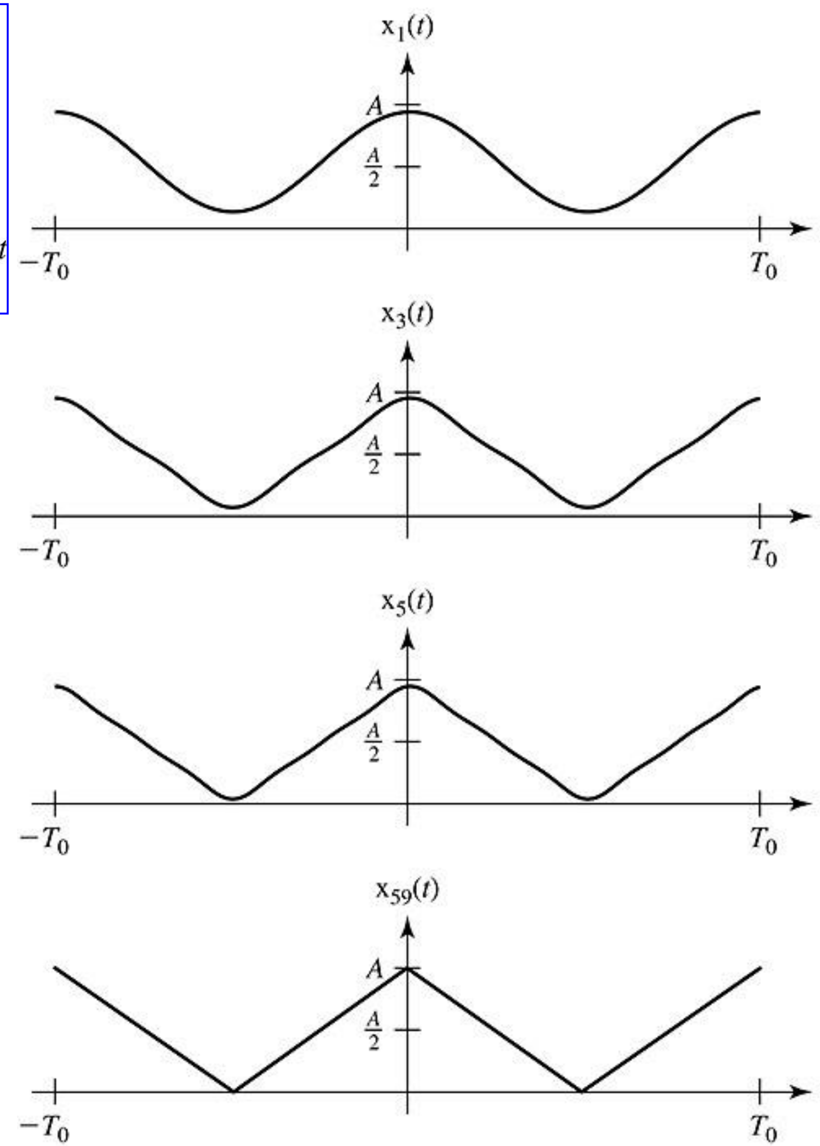






Example 3


x(t)
A
...
...
t
T0


x1(t)
A
A/2
t
−T0
T0
x3(t)
A
A/2
t
−T0
T0
x5(t)
A
A/2
t
−T0
T0
x59(t)
A
A/2
t
−T0
T0

Continuous Time Fourier Series (CTFT)

- Derivation of the formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Properties of CTFS

- There are **15** properties in Table 3.1 in page 206 of the textbook

- Do we have to remember them all?
  - No
  - Instead, familiarize yourself with them and be able to derive them whenever necessary

- They all come from the single formula

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Selected Properties

Given two periodic signals with same period *T* and fundamental frequency $\omega_0 = 2\pi/T$:

$$x(t) \leftrightarrow a_k$$
$$y(t) \leftrightarrow b_k$$

1. Linearity: $z(t) = Ax(t) + By(t) \leftrightarrow Aa_k + Bb_k$
2. Time-Shifting: $z(t) = x(t - t_0) \leftrightarrow a_k e^{-jk\omega_0 t_0}$
3. Time-Reversal (Flip): $z(t) = x(-t) \leftrightarrow a_{-k}$
4. Conjugate Symmetry: $z(t) = x^*(t) \leftrightarrow a^*_{-k}$

Selected Properties

5. x(t) is real and even → a_k is real and even

6. x(t) is real and odd → a_k is purely imaginary and odd

7. Multiplication: $$z(t) = x(t)y(t) \leftrightarrow \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

8. Parseval's Relation: $$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Other Forms of CTFS

real

Let $f(x)$ be a function of period $p = 2L$. Then, its Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

with the coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x)\, dx, \\ a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\, dx, \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x\, dx. \end{aligned}$$

Discrete Time Fourier Series

How to represent a periodic function x[n] with period N as

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n} = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

	continuous time	discrete time
periodic (series)	CTFS	**DTFS**
aperiodic (transform)	CTFT	DTFT

Discrete Time Period Functions with Period N

- $x[n]=x[n+N]$
  - Fundamental frequency $\omega_0=2\pi/N$

- $\{\phi_k[n] = e^{jk\omega_0 n} = e^{jk(2\pi/N)n}$: $k=0, \pm 1, \pm 2, \ldots\}$ is a set of signals, consisting of all discrete-time complex exponentials that are periodic with period N
  - $\phi_k[n] = \phi_k[n+N]$
  - $\phi_k[n] = \phi_{k+N}[n] = \phi_{k+2N}[n] = \ldots$
  - Only N distinct signals in the set

- Therefore, while an infinite number of complex exponentials are required in CTFS, only N complex exponentials are used in DTFS.

Discrete Time Fourier Series (DTFS)

- All functions with period N can be represented as a DTFS

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$$

$\sum_{n=\langle N \rangle}$: denotes the sum over any interval of N successive values of n.

- Derive it!

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES

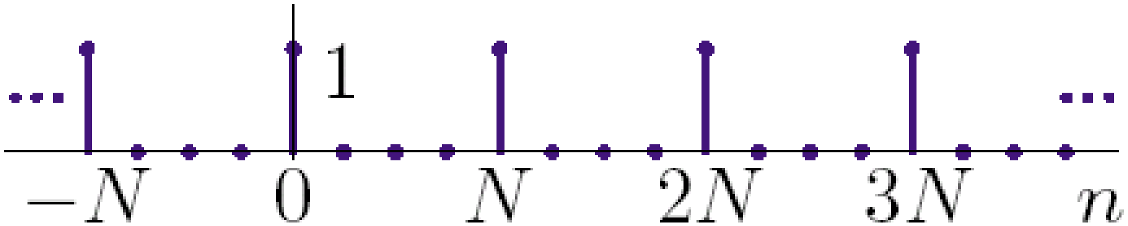
Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k$ $\begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n - r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n - 1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ $\begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

Example

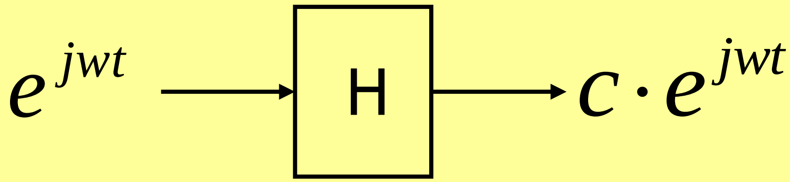
- x[n]



Fourier Series and LTI Systems

Frequency Response

CT

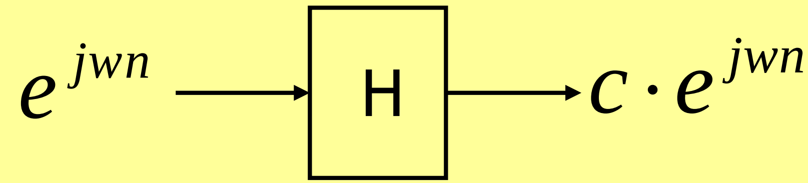


$$c = H(jw) \square \int_{-\infty}^{\infty} h(\tau) e^{-jw\tau} d\tau,$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\Rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jkw_0) e^{jk\omega_0 t}$$

DT



$$c = H(e^{jw}) \square \sum_{k=-\infty}^{\infty} h[k] e^{-jwk},$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$\Rightarrow y[n] = \sum_{k=\langle N \rangle} a_k H(e^{jk(\frac{2\pi}{N})}) e^{jk(\frac{2\pi}{N})n}$$

$H(jw)$ or $H(e^{jw})$ are called frequency responses.

Example

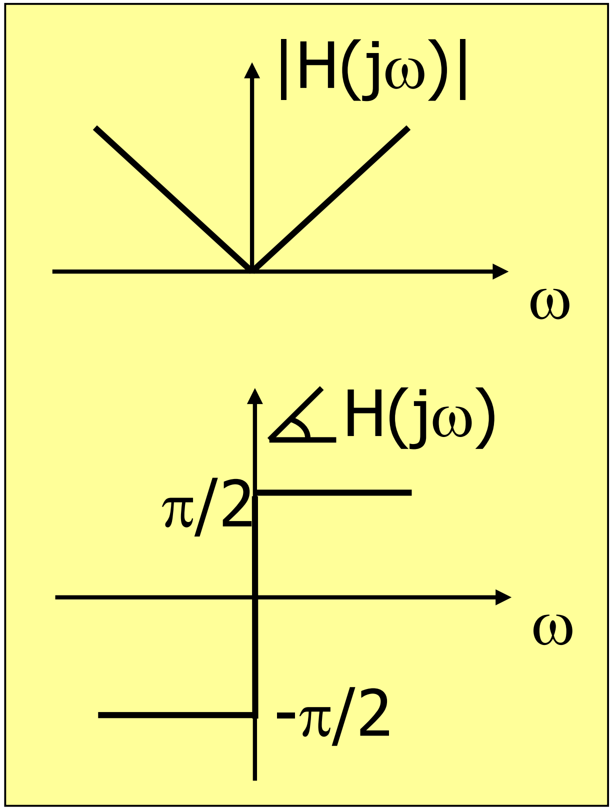
- Example 3.16 in pp. 228 in textbook

Filtering

- Filtering is a process that changes the amplitude and phase of frequency components of an input signal
  - All LTI systems can be thought as filters

- Ex) Differentiator

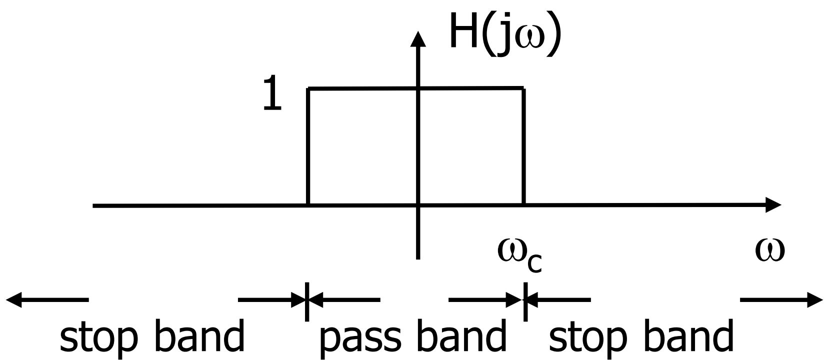
$$y(t) = \frac{dx(t)}{dt} \rightarrow H(j\omega) = j\omega = |\omega| e^{j\phi}$$



  - High frequency component is magnified, while low frequency component is suppressed
  - It is a kind of highpass filter

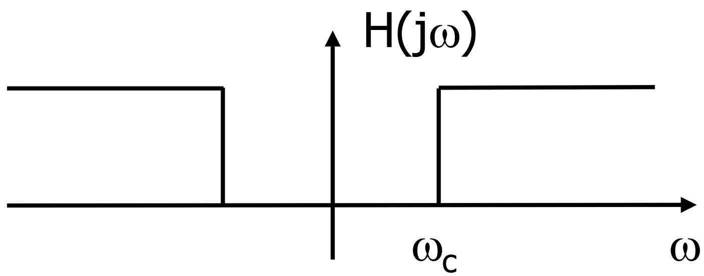
Lowpass, Bandpass and Highpass Filters

- CT

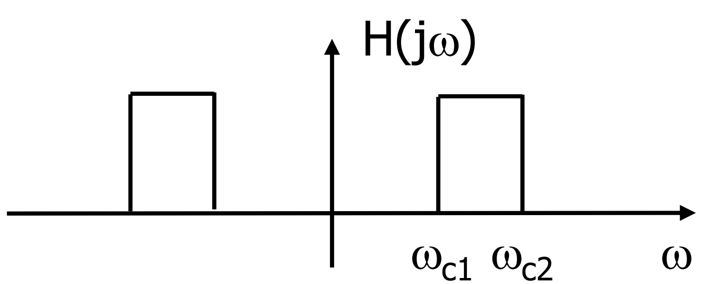


Ideal low-pass filter (LPF)

$$H(j\omega) = \begin{cases} 1 \text{ in pass band} \\ 0 \text{ in stop band} \end{cases}$$



Ideal high-pass filter (HPF)

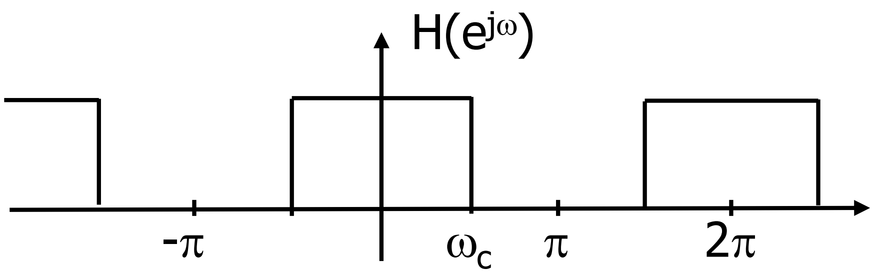


Ideal band-pass filter (BPF)

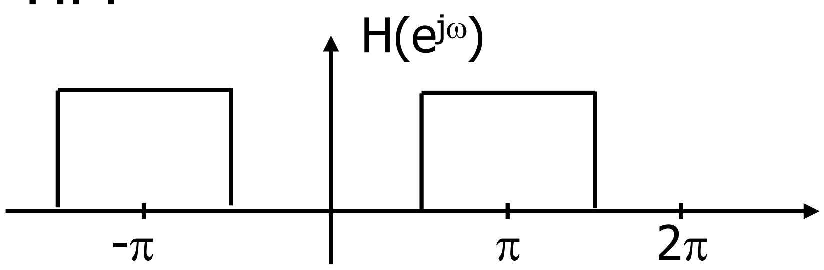
Lowpass, Bandpass and Highpass Filters

- DT

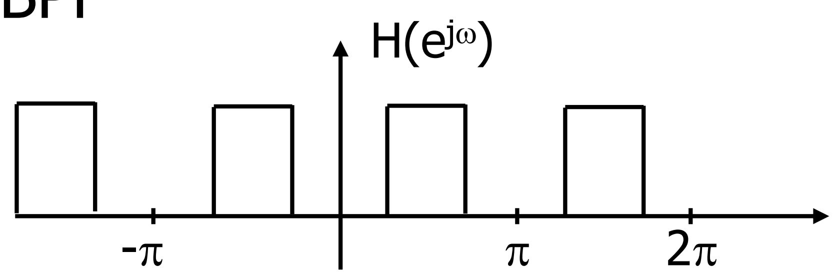
LPF



HPF



BPF

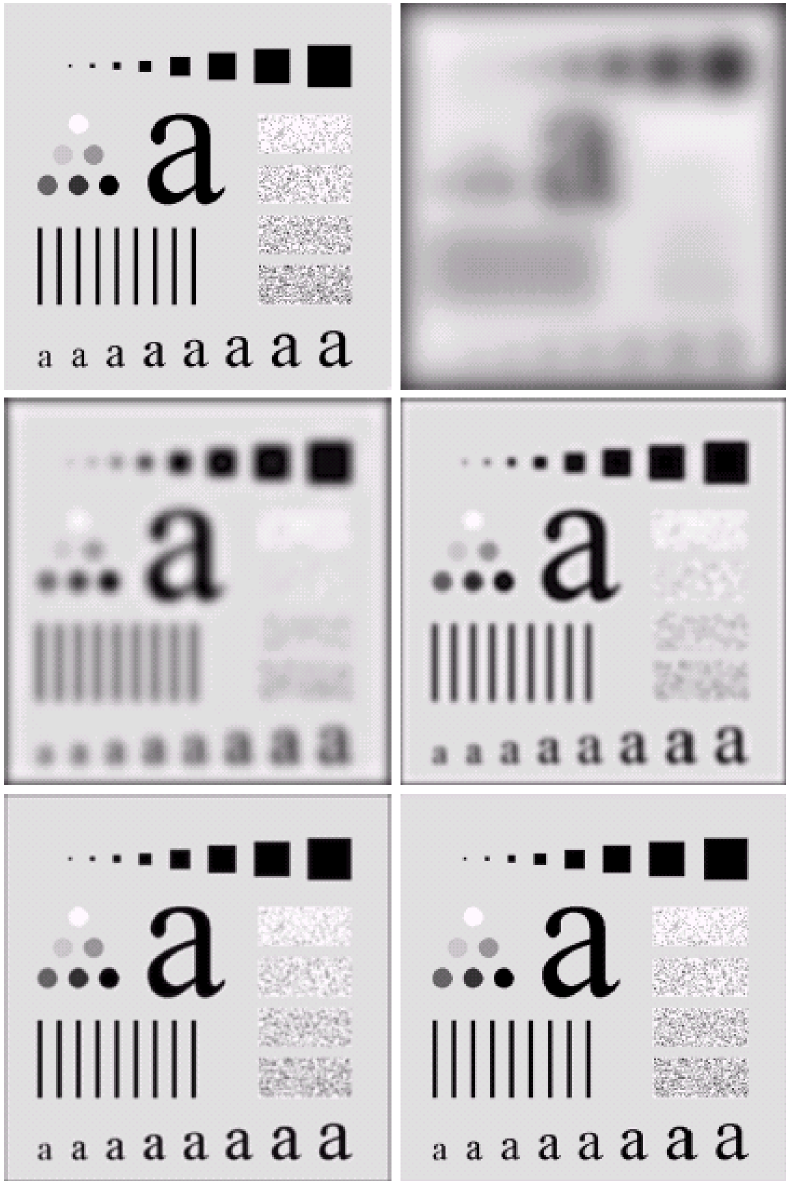


$H(e^{j\omega})$ is periodic with period 2π

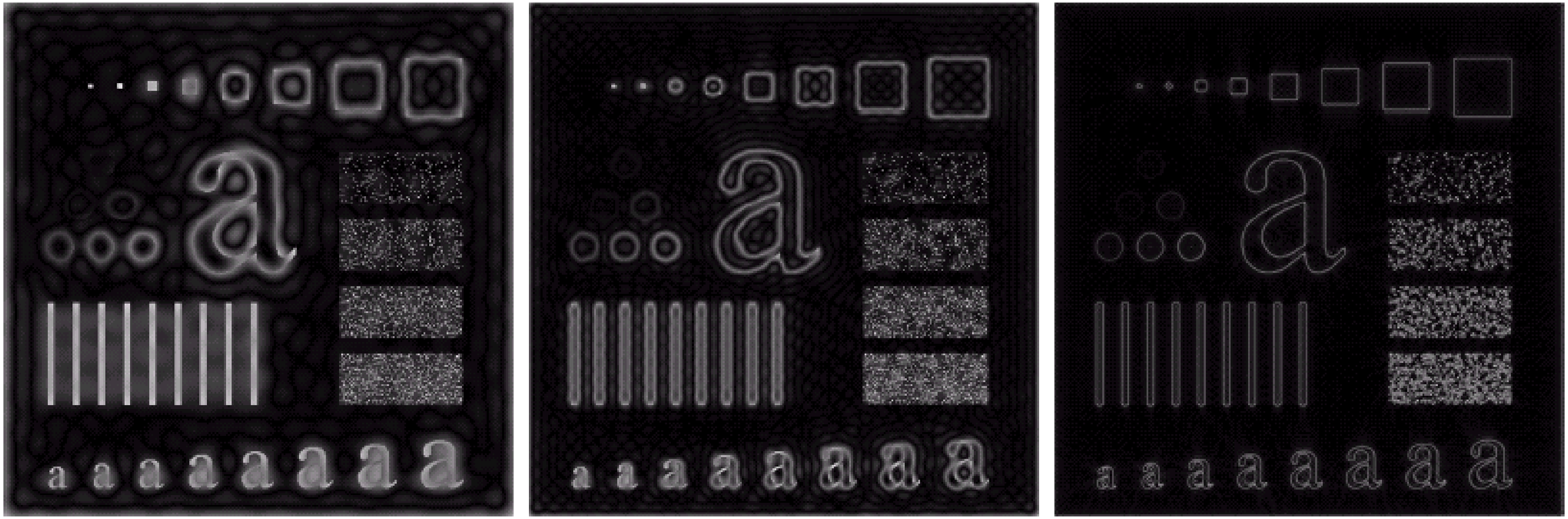
low frequencies: at around $\omega=0, \pm 2\pi$,...

high frequencies: at around $\omega= \pm\pi$, $\pm 3\pi$, ...

Lowpass Filtering



Highpass Filtering

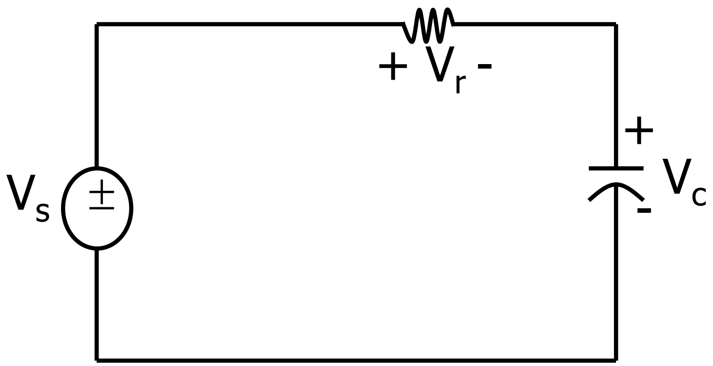


a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

Example of CT Filter

- RC lowpass filter



$$RC\frac{dV_c(t)}{dt}+V_c(t)=V_s(t)$$

$$\text{input}: V_s(t)$$

$$\text{output}: V_c(t)$$

$$\therefore RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right]+H(j\omega)e^{j\omega t}=e^{j\omega t}$$

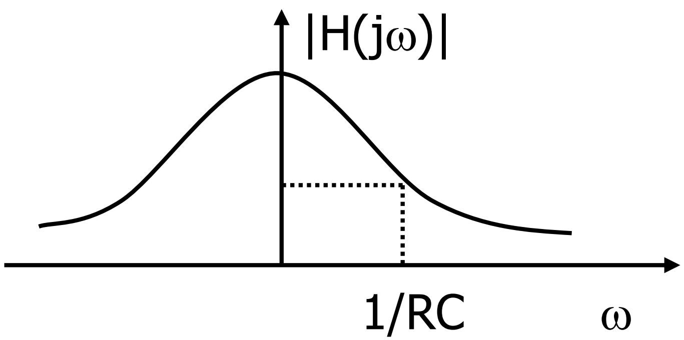
$$\therefore RC\cdot j\omega\cdot H(j\omega)e^{j\omega t}+H(j\omega)e^{j\omega t}=e^{j\omega t}$$

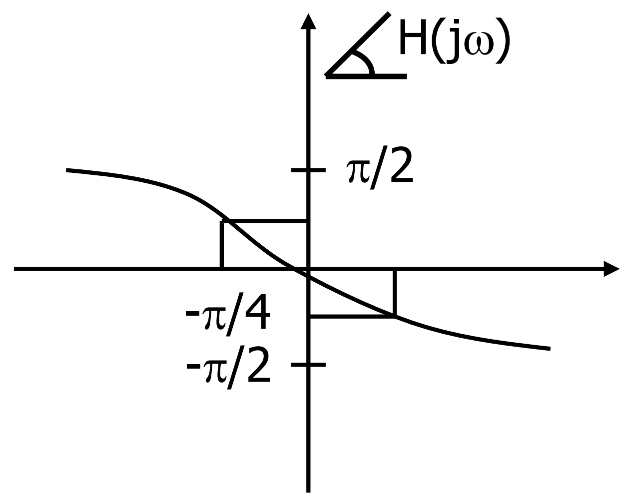
$$\rightarrow H(j\omega)=\frac{1}{1+RCj\omega}=\frac{1}{\sqrt{1+(RC\omega)^2}}e^{j\tan^{-1}(-RC\omega)}$$

Example of CT Filter (continued)

- It is a lowpass filter

$$H(j\omega) = \frac{1}{1 + RCj\omega} = \frac{1}{\sqrt{1 + (RC\omega)^2}} e^{j \tan^{-1}(-RC\omega)}$$





Example of DT Filter 1

- $y[n] - ay[n-1] = x[n]$

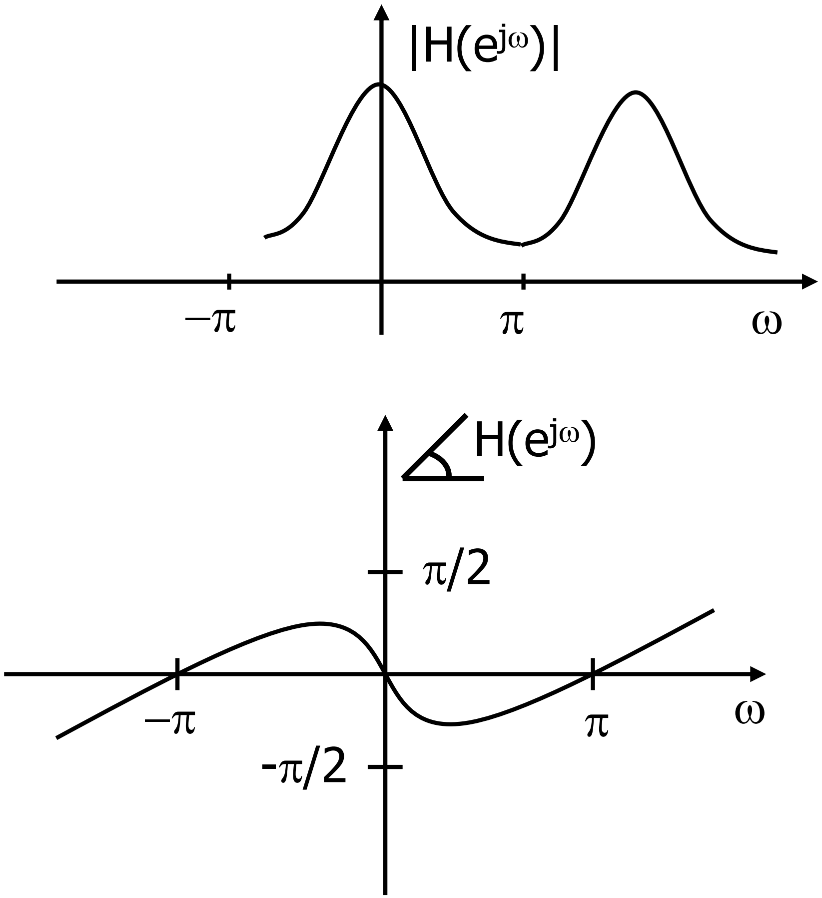
We know if $x[n]=e^{j\omega n}$ then $y[n]=H(e^{j\omega})e^{j\omega n}$

$$\rightarrow H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

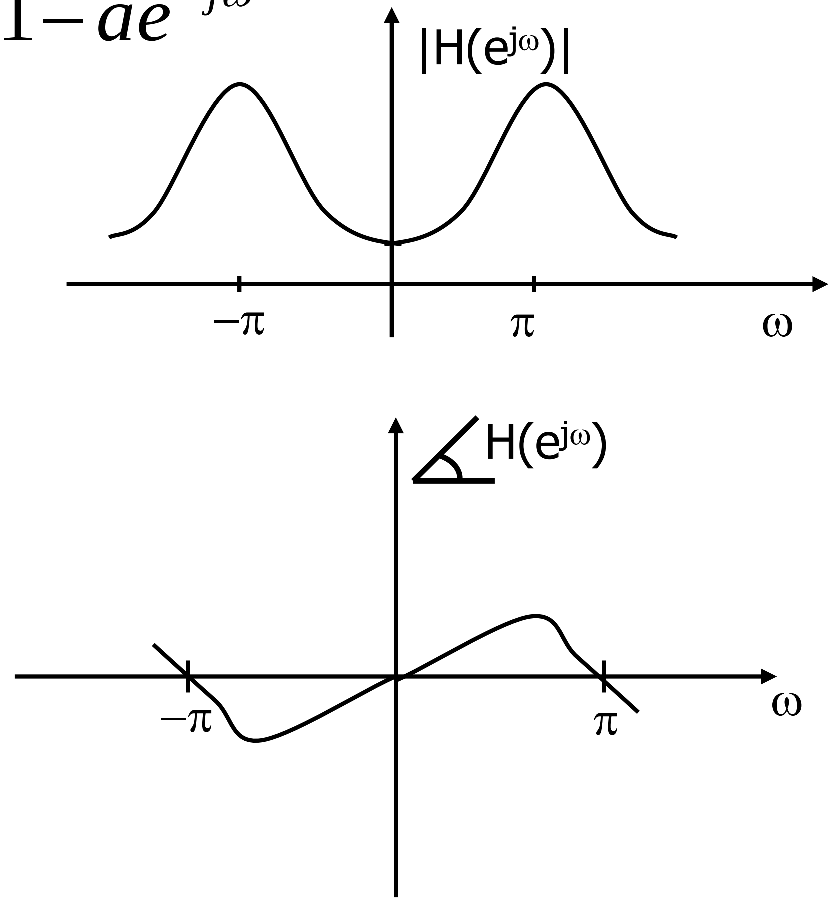
$$\rightarrow H(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

Example of DT Filter 1 (Continued)

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



(a) a=0.6 ⇒ LPF



(b) a=-0.6 ⇒ HPF

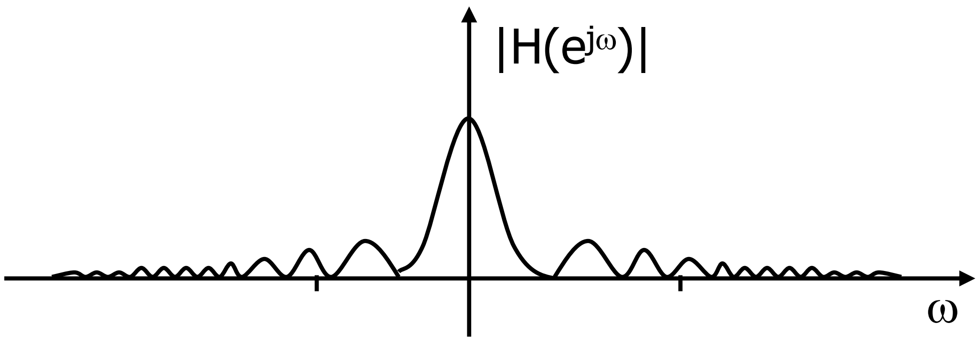
Example of DT Filter 2

- $$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]$$

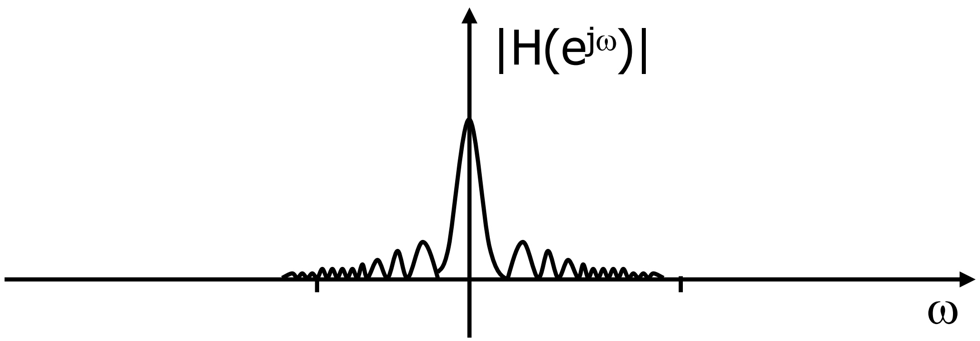
$$h[n] = \begin{cases} \frac{1}{N+M+1}, & for \quad -N \le n \le M \\ 0, & otherwise \end{cases}$$

$$\Rightarrow H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-j\omega k}$$

Example of DT Filter 2 (Continued)



a) N=M=16



b) N=M=32

It is a lowpass filter