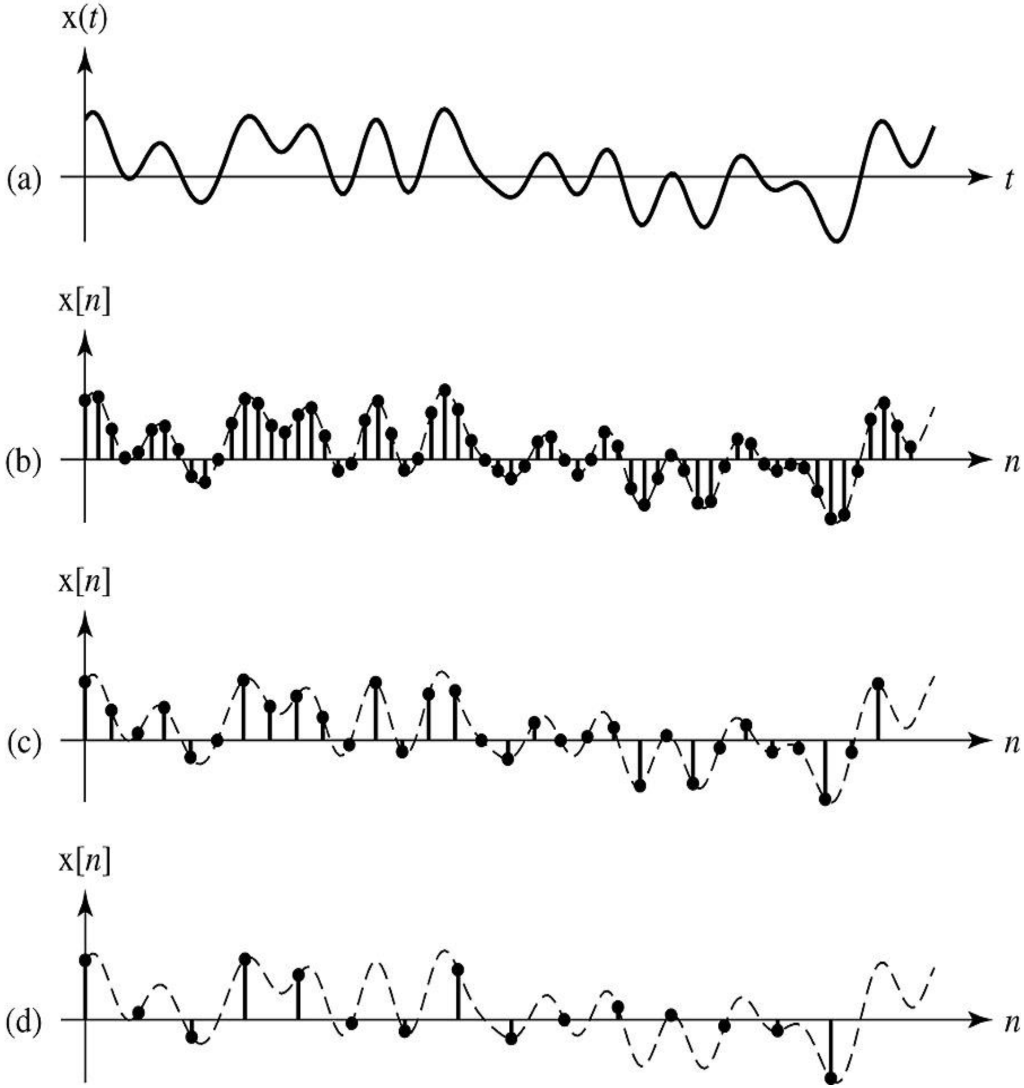
Sampling

Chang-Su Kim

Some figures have been excerpted from

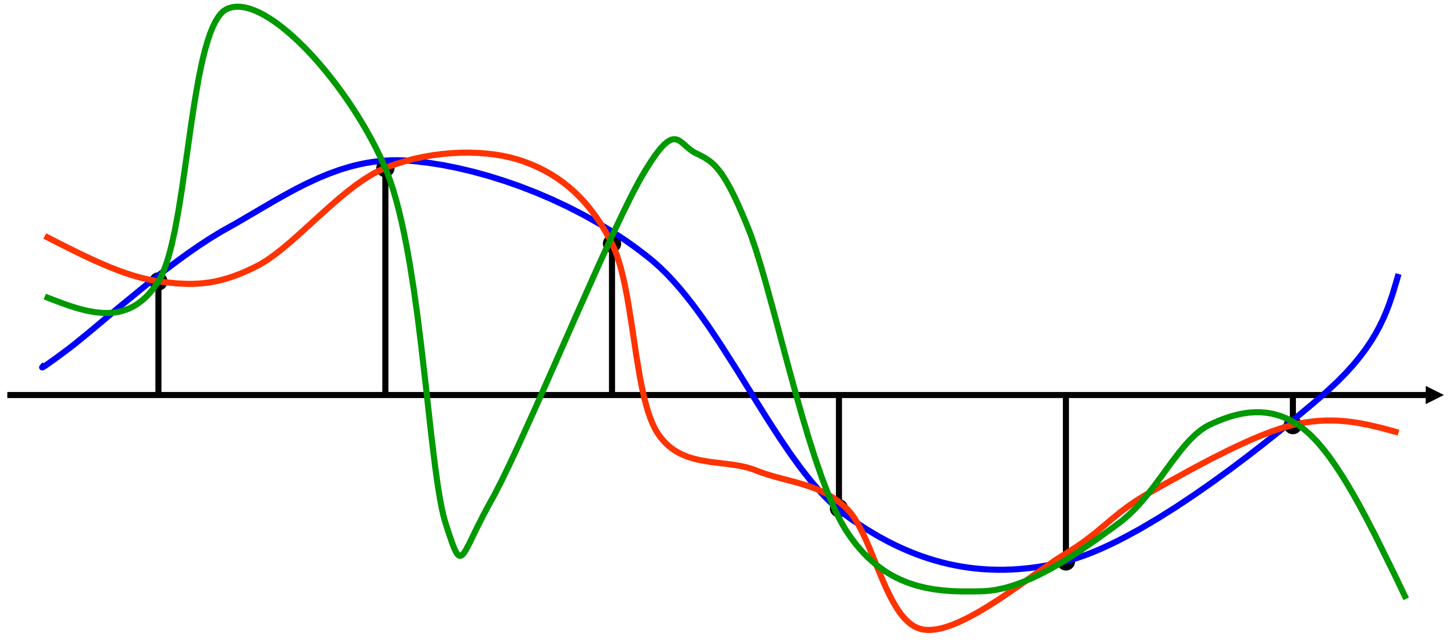
1. The lecture notes of Dr. Benoit Boulet in McGill University (http://www.cim.mcgill.ca/~boulet/304-304A/304-304A.htm)
2. *Signals and Systems* by M. J. Roberts
3. http://www.ics.uci.edu/~majumder/CG/classes/sampling_1nov.pdf

Sampling



- Sampling is a procedure to extract a DT signal from a CT signals
- (b), (c), (d) are obtained by sampling (a)
- Is (b) enough to represent (a)?
- What is the adequate sampling rate to represent a given CT signal without information loss?

In general, DT signal cannot represent CT signal perfectly



Are these sample enough to reconstruct the original blue curve?

Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(jw) = 0$ for $|w| > w_M$. Then, $x(t)$ is uniquely determined by its samples $x(nT)$, if

$$w_s > 2\, w_M$$

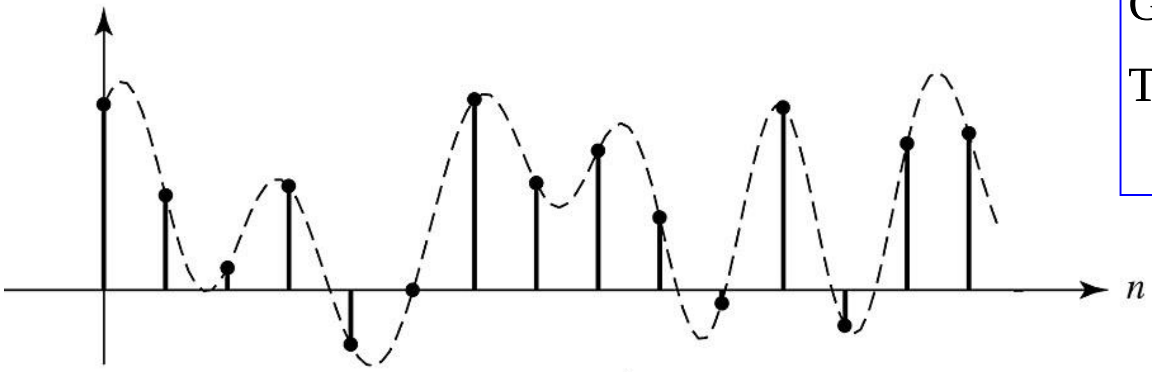
where the sampling rate w_s is defined as

$$w_s = 2\pi/T$$

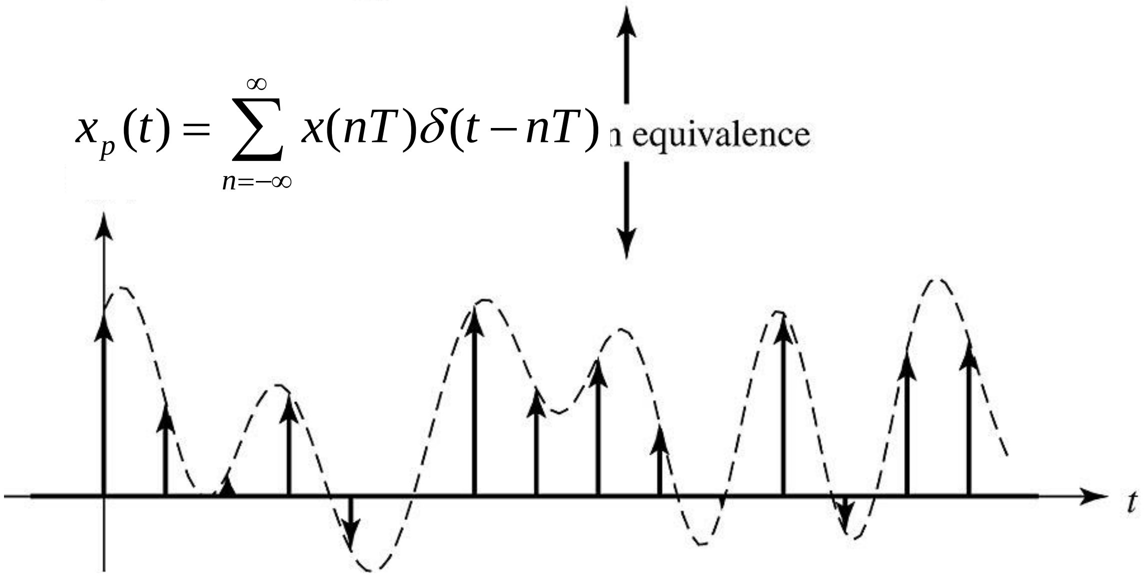
- Under certain conditions, a CT signal can be completely represented by and recoverable from samples
- A lowpass signal can be reconstructed from samples, if the sampling rate is high enough. Because it is a lowpass signal, the change between two close samples is constrained (or expected).

Information Equivalence

$$x[n] = x(nT)$$



$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



Given $x[n]$, we can generate $x_p(t)$.

Given $x_p(t)$, we can generate $x[n]$.

Therefore, $x[n]$ and $x_p(t)$ have the same information.

Restated Sampling Theorem

Let $x(t)$ be a band-limited signal with $X(jw) = 0$ for $|w| > w_M$. Then, $x(t)$ is uniquely determined by the modulated impulse train

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

if

$$w_s > 2\, w_M$$

where the sampling rate w_s is defined as

$$w_s = 2\pi/T$$

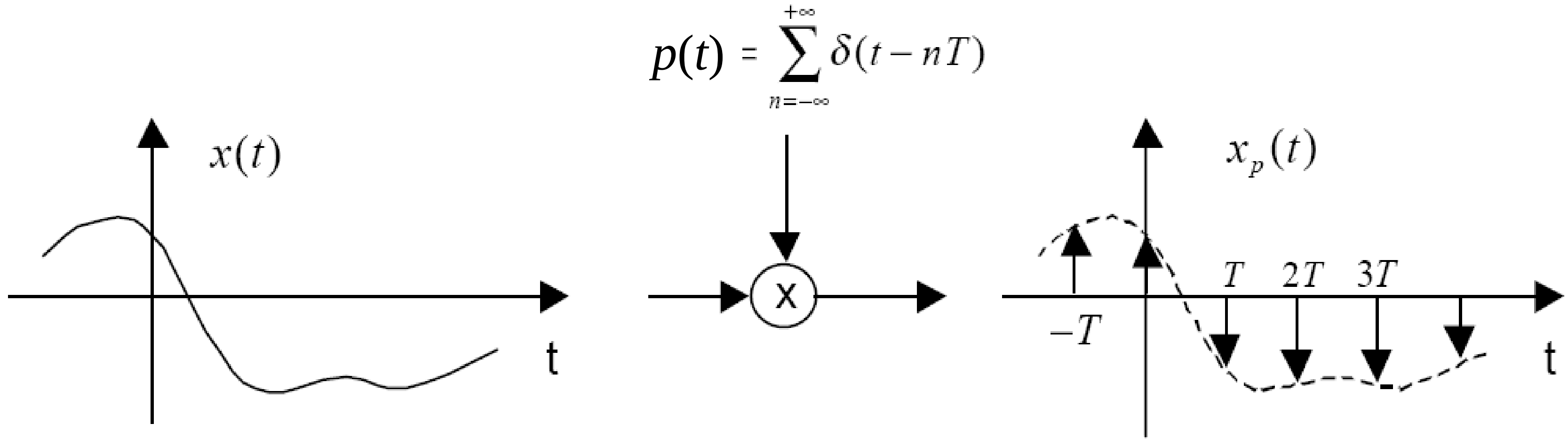
$2\, w_M$: Nyquist rate

Frequency Domain Interpretation

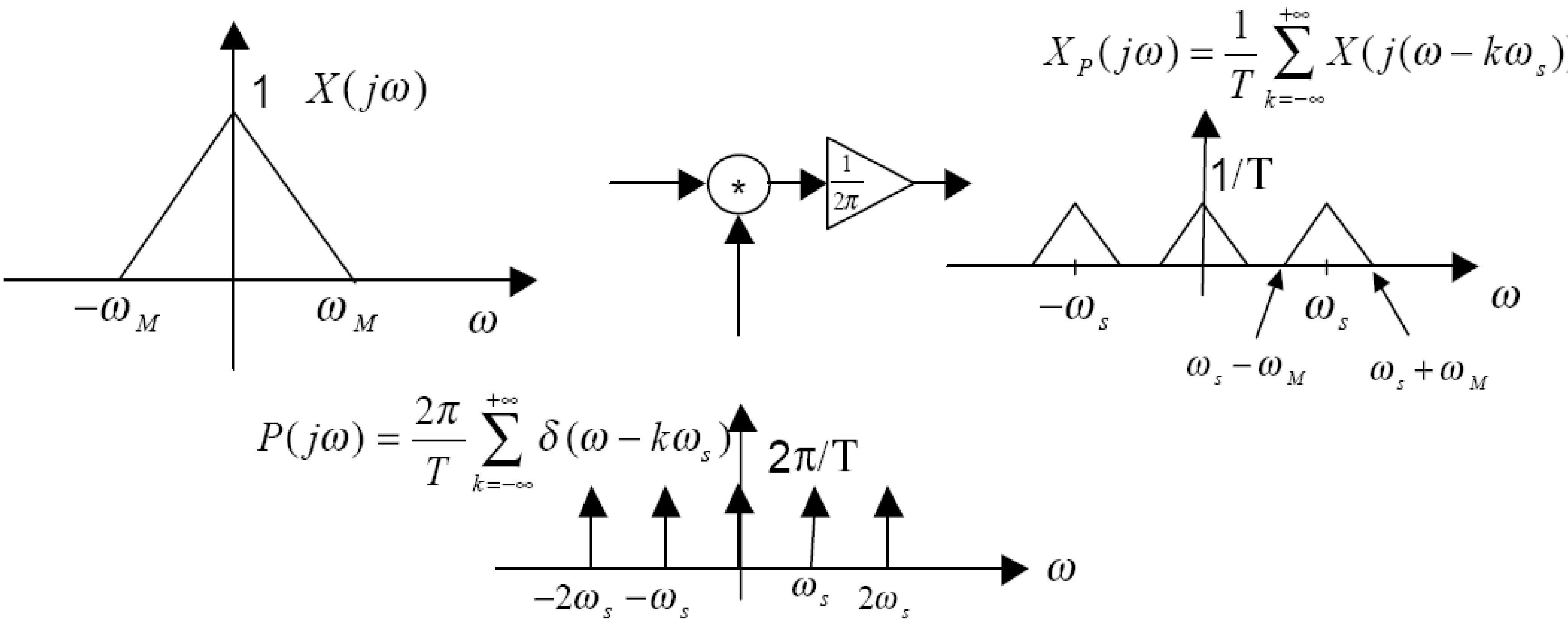
Show that

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

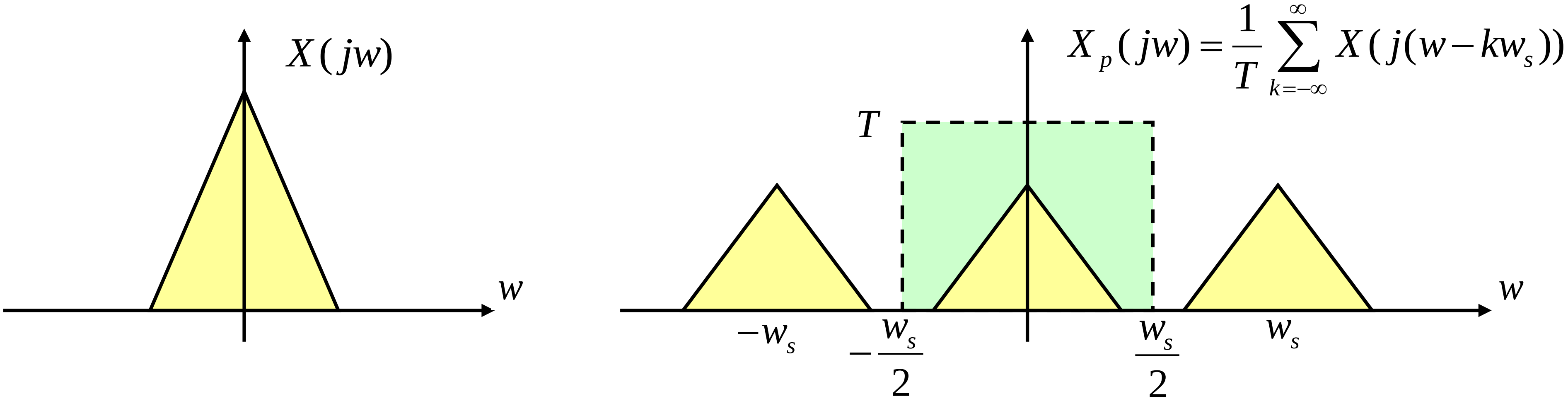
$$\Rightarrow X_p(jw) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(w-kw_s))$$



In the frequency domain:



Reconstruction of Signal from Its Samples



Reconstruction: ideal lowpass filter

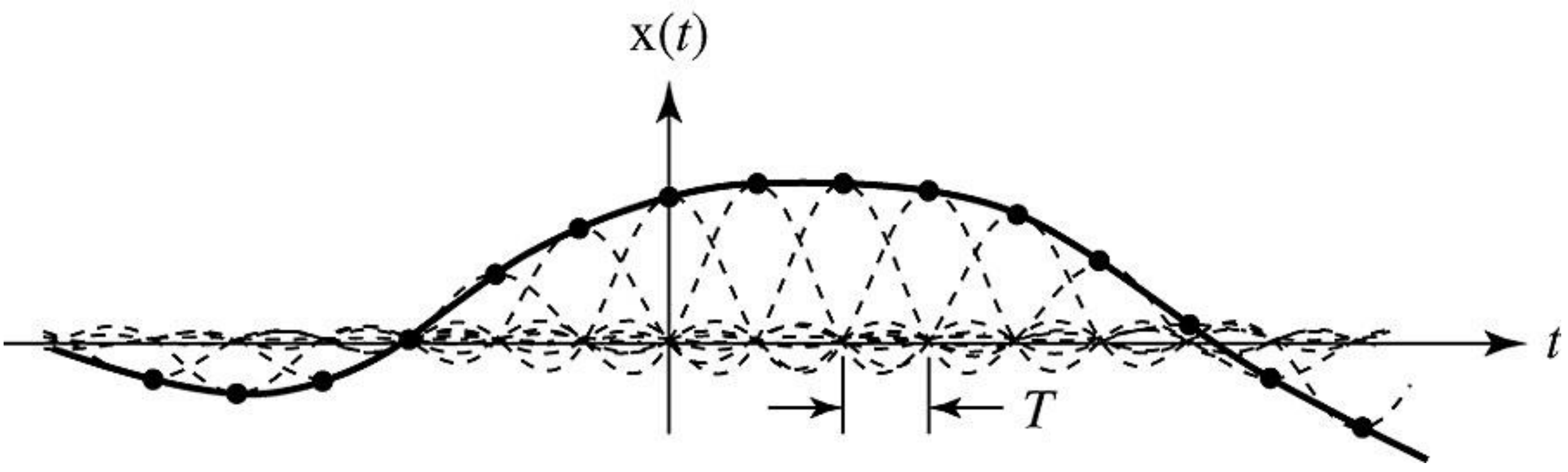
$$H(jw) = \begin{cases} T & |w| < \frac{w_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow h(t) = T\frac{\sin\frac{w_s}{2}t}{\pi t} = \frac{\sin\frac{\pi}{T}t}{\frac{\pi}{T}t} = \text{sinc}(\frac{t}{T})$$

Reconstruction of Signal from Its Samples

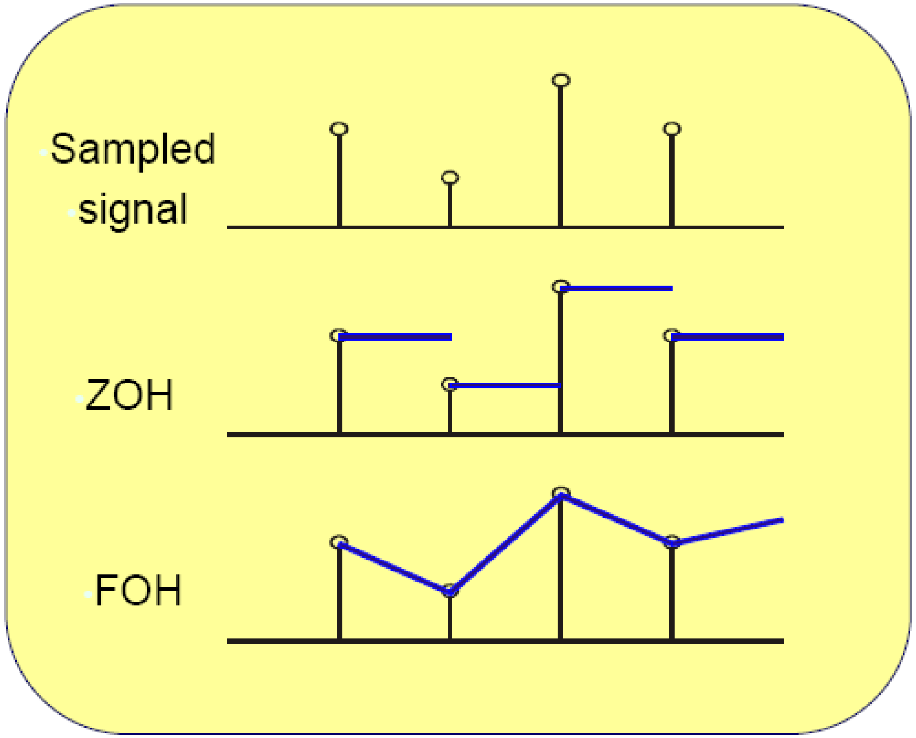
$$x_r(t) = h(t) * x_p(t)$$
$$= \text{sinc}(t/T) * \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$
$$= \sum_{n=-\infty}^{\infty} x(nT)\,\text{sinc}\left(\frac{t-nT}{T}\right)$$

- This is the way to reconstruct or interpolate $x_r(t)$ from samples x(nT)'s
- Note that $x_r(t) = x(t)$, if the sampling rate is higher than the Nyquist rate

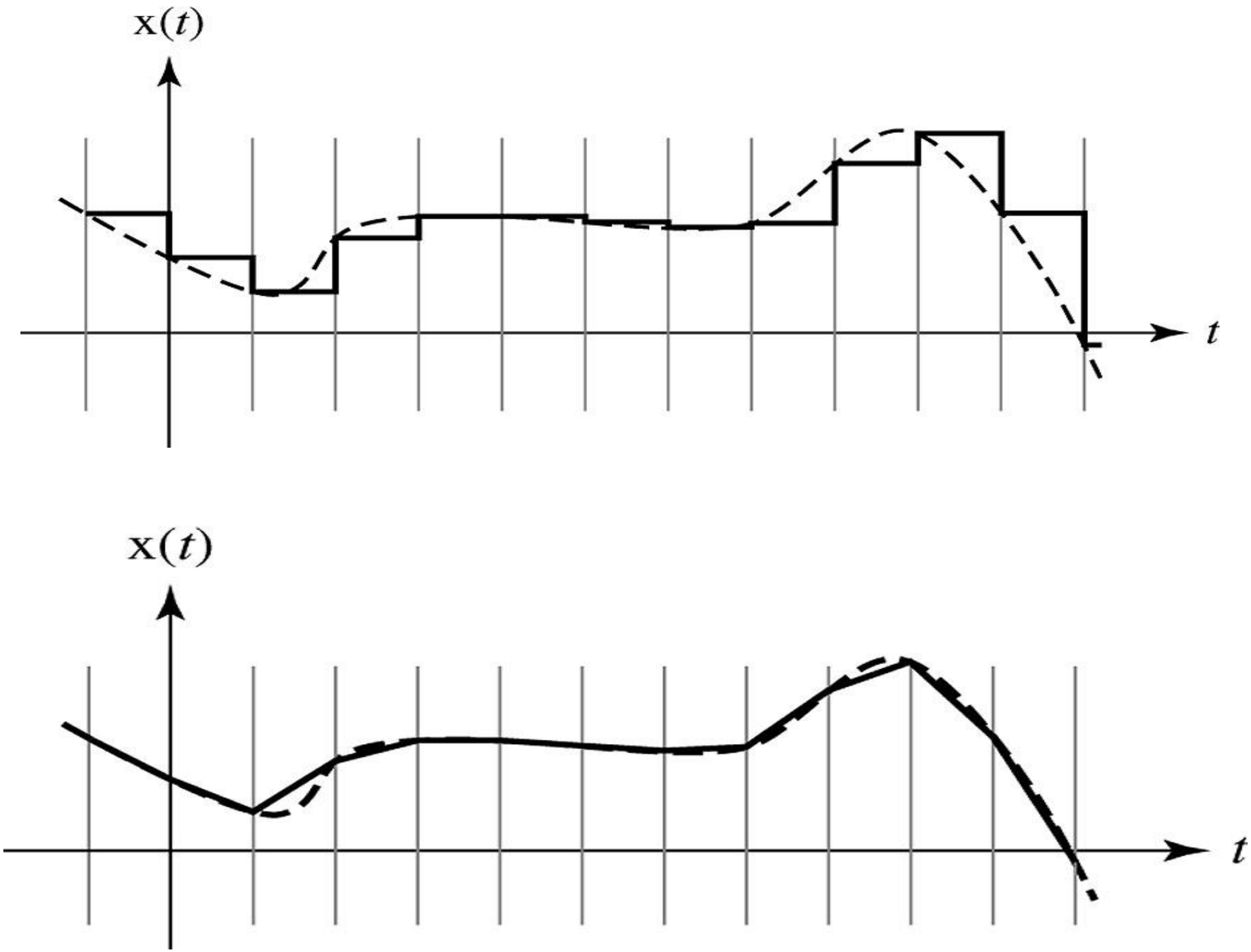


Practical Interpolation Filters

- Ideal interpolation filters – sinc function
  - Infinite duration
  - Not implementable
- Practical interpolation filters
  - Zero order holding (repetition)
  - First order holding (linear)

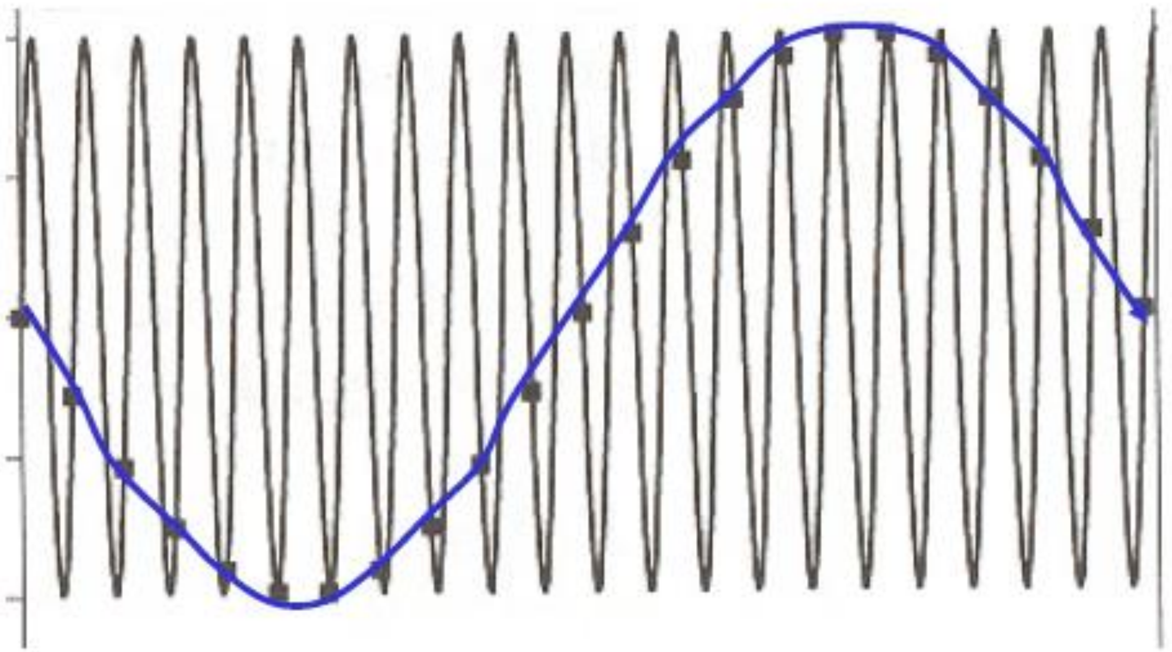


Practical Interpolation Filters

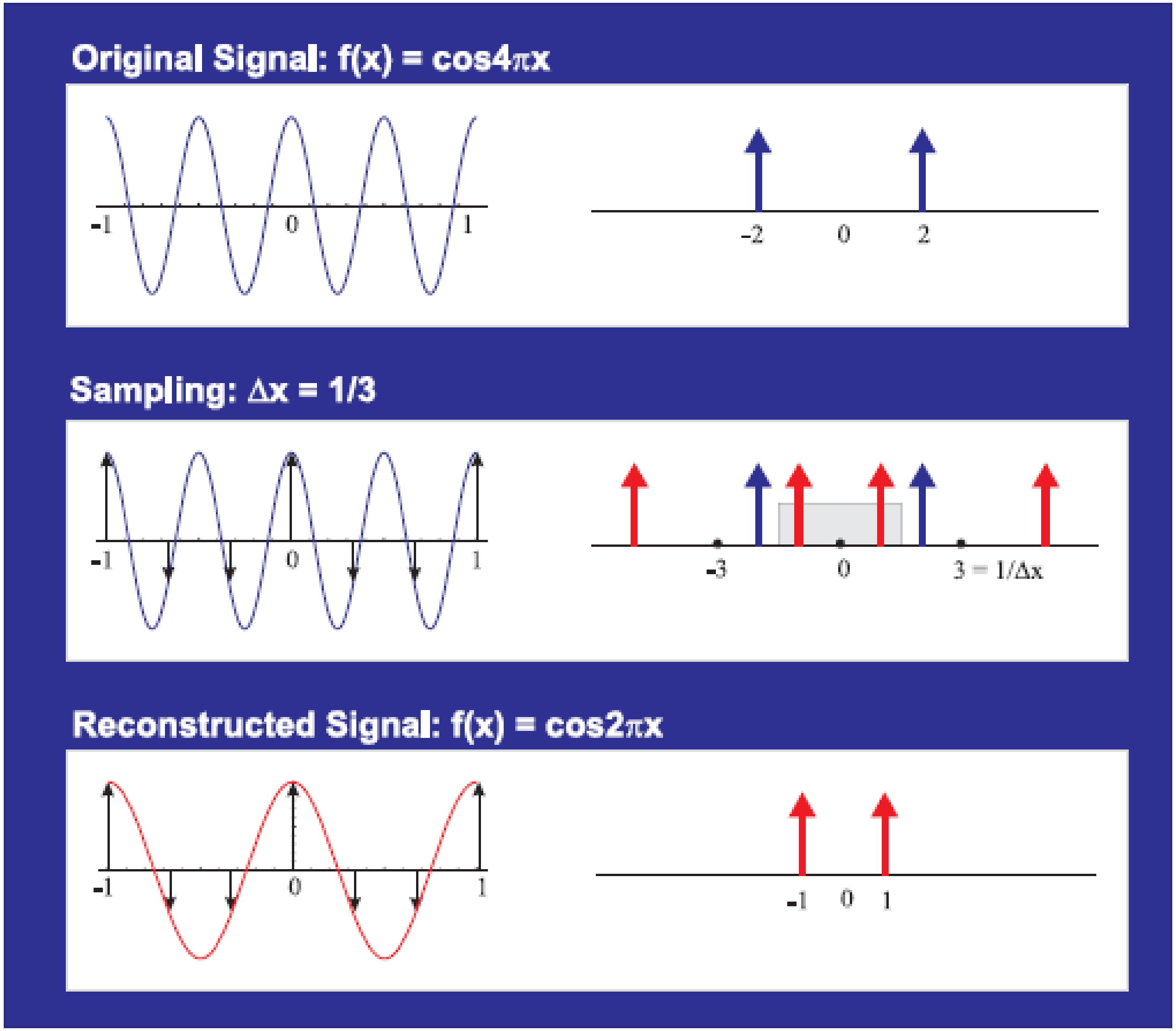


Undersampling Causes Aliasing

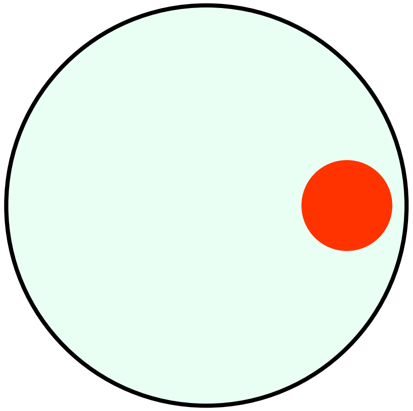
- Undersampling: sampling rate is less than Nyquist rate



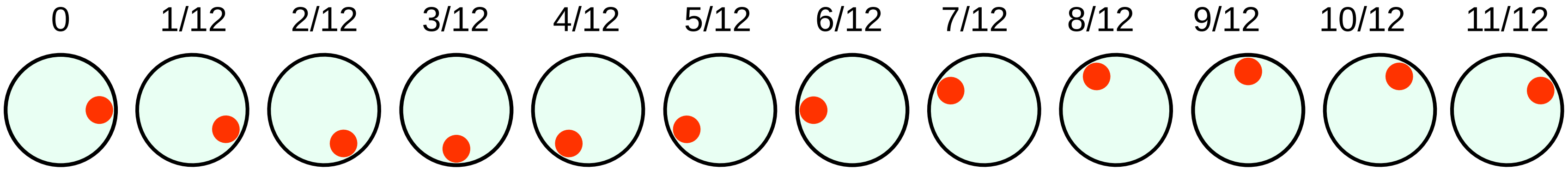
Undersampling Causes Aliasing



Undersampling Causes Aliasing

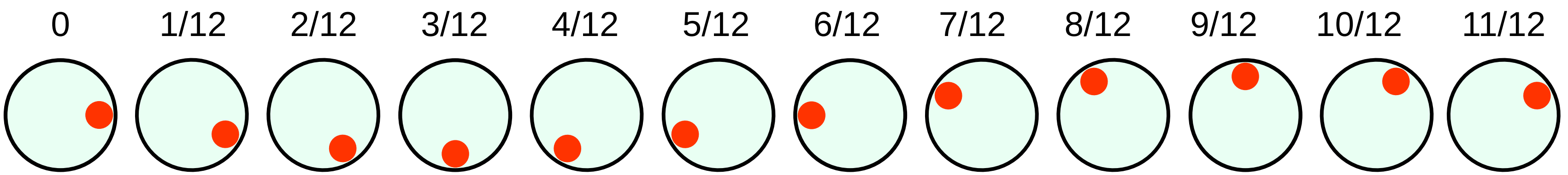


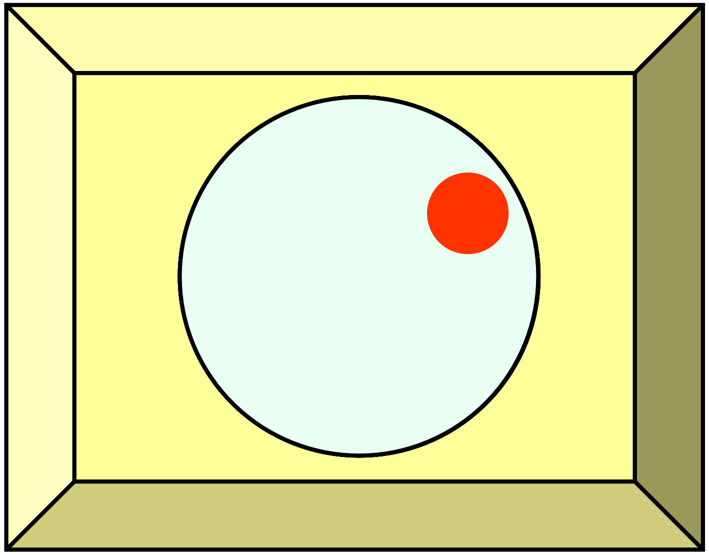
- Rotating disk
  - 1 rotation/second
- To avoid aliasing, it should be motion-pictured with at least 2 frames/s.



Undersampling Causes Aliasing

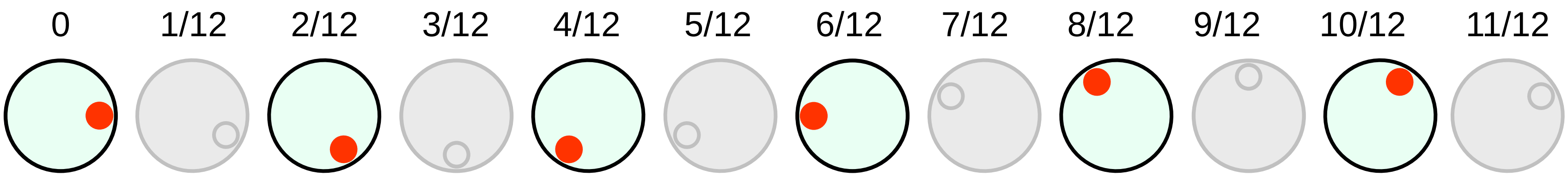
- 12 frames/s

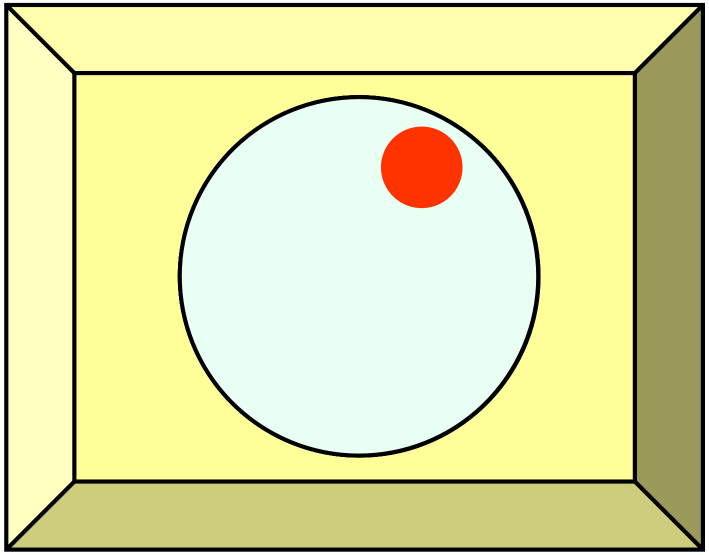




Undersampling Causes Aliasing

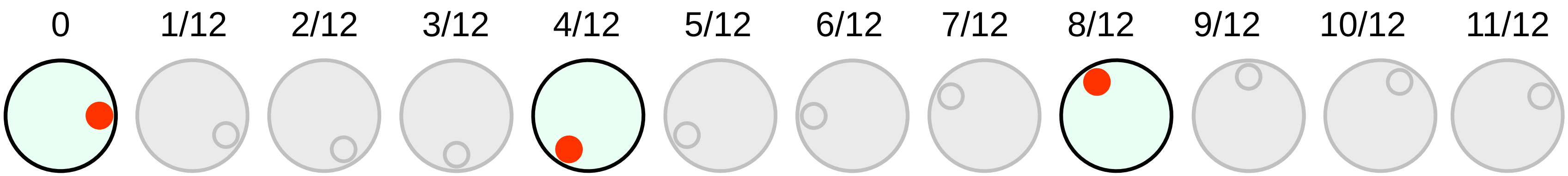
- 6 frames/s

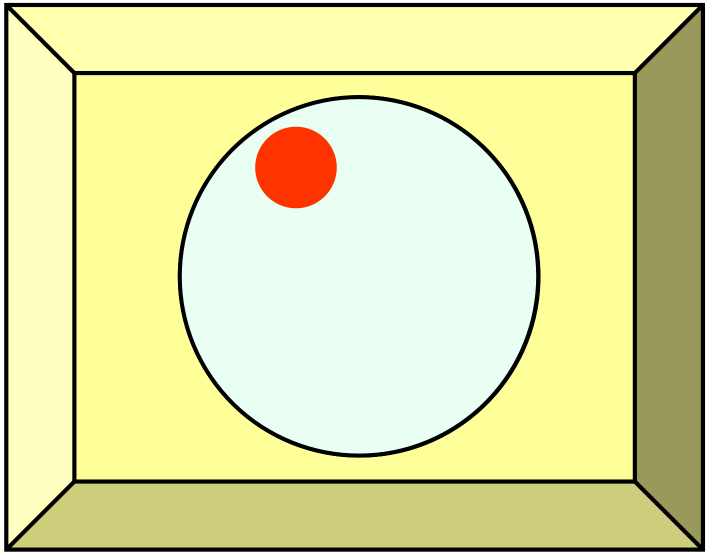




Undersampling Causes Aliasing

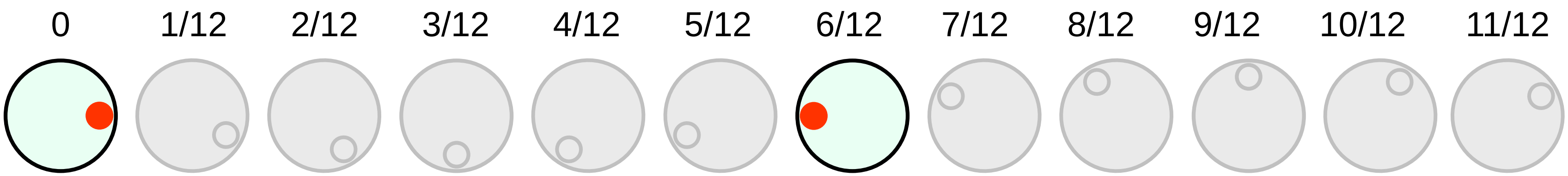
- 3 frames/s

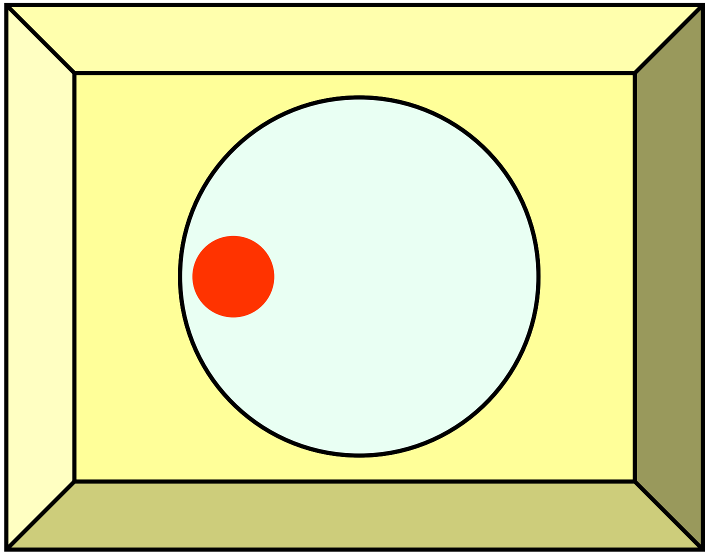




Undersampling Causes Aliasing

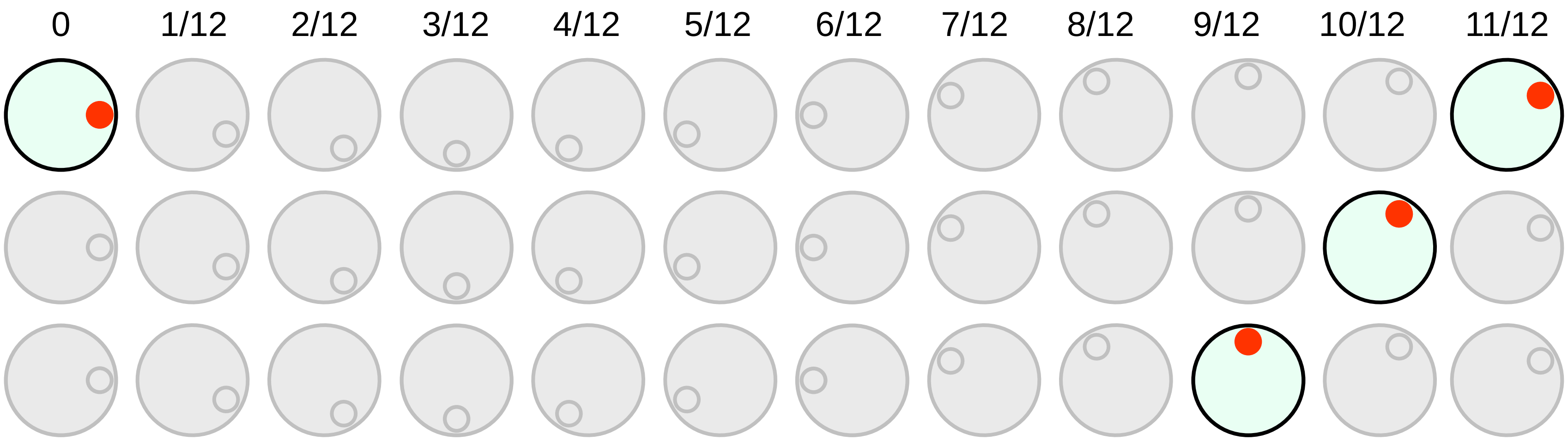
- 2 frames/s

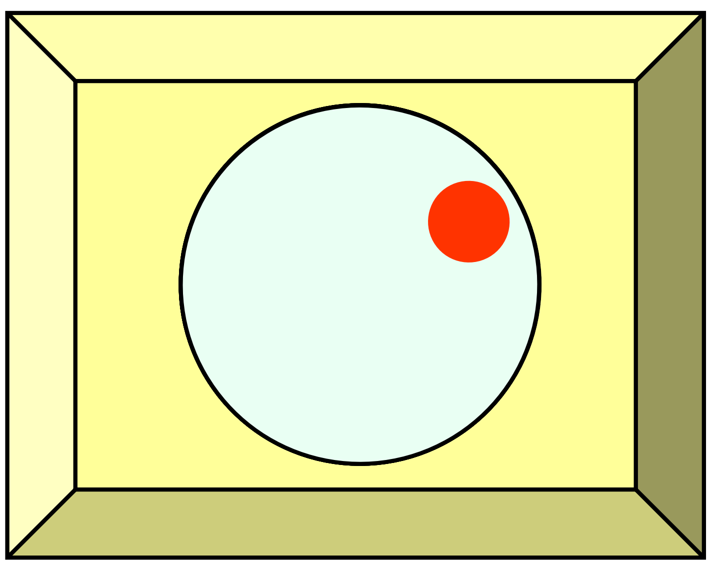




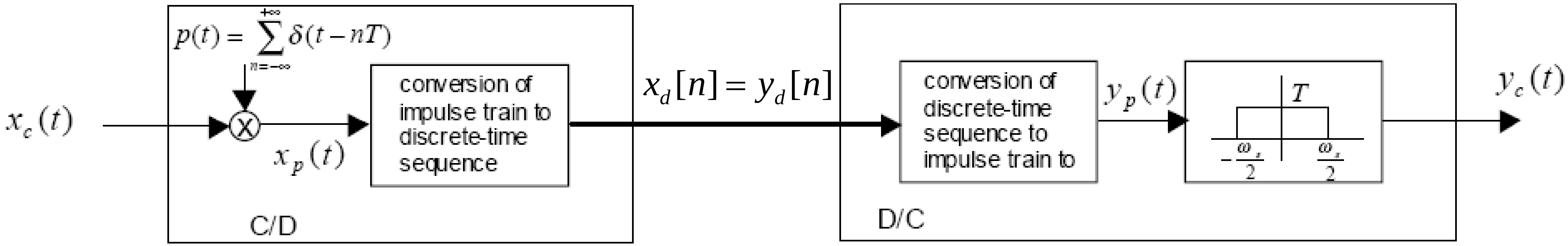
Undersampling Causes Aliasing

- 12/11 = 1.09 frames/s



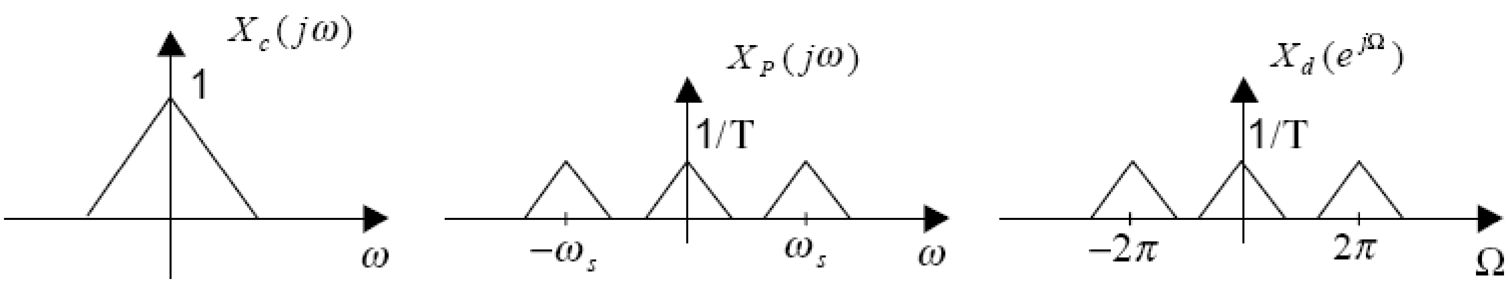


DT Processing of CT Signals

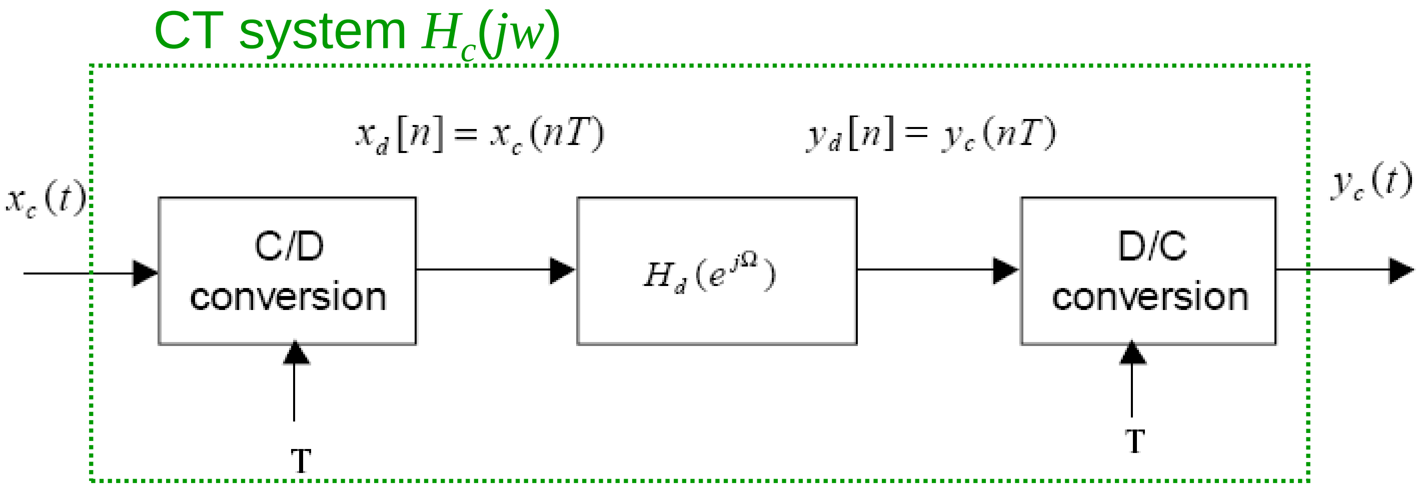


- We assume that the sampling rate is higher than Nyquist rate
- So $y_c(t) = x_c(t)$
- Relation between modulated impulse train and DT signal

$$X_d(e^{j\Omega}) = X_p(j\Omega/T) \text{ or } X_p(jw) = X_d(e^{jwT}),$$
$$Y_d(e^{j\Omega}) = Y_p(j\Omega/T) \text{ or } Y_p(jw) = Y_d(e^{jwT})$$



DT Processing of CT Signals



$$Y_c(jw) = X_c(jw) H_d(e^{jwT})$$

$$\text{or } H_c(jw) = H_d(e^{jwT})$$

This is true if $x_c(t)$ is band-limited and T satisfies the Nyquist condition

DT Processing of CT Signals

- Refer to Figure 7.25