The z-Transform

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Some figures have been excerpted from
- The lecture notes of Dr. Weiss in MIT (http://umech.mit.edu/weiss/lectures.html)
z-Transform is an extension of DTFT

- **z-Transform**
  
  \[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

- **DTFT**
  
  \[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \]

- **z-Transform vs. DTFT**
  
  \[ X(z) \big|_{z=e^{j\omega}} = \text{DTFT of } x[n] \]
Why do we need the extension?

- Consider the DTFT pair

\[ a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1 - ae^{-jw}} \quad |a| < 1 \]

  - What happens if $|a| \geq 1$?

- z-Transform pair

\[ a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \quad |z| > |a| \]

  - ROC (region of convergence)

- z-Transform can be applied to a broader class of signals than DTFT
  - It is useful in studying a broader class of systems
  - It is used to analyze the causality and stability of a system
Inverse z-Transform

\[ x[n] = \frac{1}{2\pi j} \int X(z) z^{n-1} \, dz \]

- Integral along a circular path in the complex number plane
  - Its proper evaluation requires some knowledge on complex integral
  - For example, refer to
- We do not use this formula. Instead, we decompose \( X(z) \) into a number of terms, each of which can be inverse transformed using Table 10.2.
## Parts of Table 10.2

<table>
<thead>
<tr>
<th>z-Transform Pair</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $\delta[n-m]$ $\iff$ $z^{-m}$</td>
<td>All $z$ except $z= 0$ or $\infty$</td>
</tr>
<tr>
<td>(2) $\alpha^n u[n]$ $\iff$ $\frac{1}{1 - \alpha z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>(3) $-\alpha^n u[-n-1]$ $\iff$ $\frac{1}{1 - \alpha z^{-1}}$</td>
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<tr>
<td>(4) $[\cos \omega_0 n] u[n]$ $\iff$ $\frac{1 - [\cos \omega_0] z^{-1}}{1 - (2 \cos \omega_0) z^{-1} + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>(5) $[\sin \omega_0 n] u[n]$ $\iff$ $\frac{[\sin \omega_0] z^{-1}}{1 - [2 \cos \omega_0] z^{-1} + z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>
ROC should be specified

$$a^n u[n] \xlongequal{z} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n-1] \xlongequal{z} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$
Ex) \( x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n] \)

\[ X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| > \frac{1}{2} \]

There are other sequences, which generate the same \( X(z) \) but with different ROC’s.
ROC should be specified

Ex) $x[n] = ?$

$$X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$
**ROC should be specified**

**Ex)** \( x[n] = ? \)

\[
X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| < \frac{1}{3}
\]
Properties on ROC

- ROC of $X(z)$ consists of a single ring in the $z$-plane centered at the origin.

- ROC does not contain any poles.

- Suppose that $X(z)$ is rational.
  1. Its ROC is bounded by poles.
  2. If $x[n]$ is right sided, ROC is the region in the $z$-plane outside the outermost pole.
  3. If $x[n]$ is left sided, ROC is the region inside the innermost nonzero pole.
Selected Properties of Z-Transform

- $z^n$ is the eigenfunction of LTI systems

If $x[n] = z^n$, $y[n] = H(z)z^n$.

$H(z)$ is called system function or transfer function.
Selected Properties of Z-Transform

- $z^{-1}$ is a delay system

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z)$$
Selected Properties of Z-Transform

- Time reversal

\[ x[n] \leftrightarrow X(z), \text{ with ROC} = \mathbb{R} \]

\[ x[-n] \leftrightarrow X \left( \frac{1}{z} \right), \text{ with ROC} = \frac{1}{\mathbb{R}} \]
Selected Properties of Z-Transform

- Convolution

\[ x_1[n] \xleftarrow{} X_1(z), \text{ with ROC } = R_1 \]
\[ x_2[n] \xleftarrow{} X_2(z), \text{ with ROC } = R_2 \]
\[ x_1[n] \ast x_2[n] \xleftarrow{} X_1(z)X_2(z), \text{ with ROC containing } R_1 \cap R_2 \]
Characterization of LTI Systems Using $z$-Transform

- **Causality**
  - if the ROC of the transfer function is the exterior of a circle, including infinity

- **Stability**
  - if the ROC of the transfer function contains the unit circle $|z|=1$. 
Ex) $y[n] + \frac{1}{4} y[n-1] - \frac{1}{8} y[n-2] = x[n] - \frac{7}{4} x[n-1] - \frac{1}{2} x[n-2]$