

Digital Signal Processing

Chap 3. The z -Transform

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Definitions

z-Transform

- z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Ex) $x[n] = \delta[n + 1] + 2\delta[n] - 3\delta[n - 2]$

$$\Rightarrow X(z) = z + 2 - 3z^{-2}$$

- z-Transform is simply an alternative representation of a signal
 - The coefficient of z^{-n} is the signal value $x[n]$

z-Transform

- Ex) z-transform pair

$$\frac{1}{2^n} u[n] \xleftrightarrow{z} \frac{1}{1 - \frac{1}{2} z^{-1}},$$

$$|z| > \frac{1}{2}$$

ROC
(region of convergence)

z -Transform is an extension of DTFT

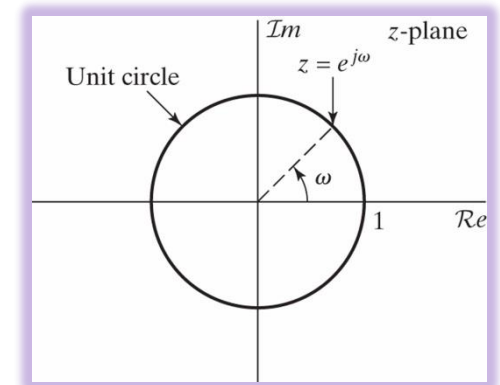
- z -Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- z -Transform vs. DTFT
 - DTFT of $x[n] = X(z)|_{z=e^{j\omega}}$



z-Transform and DTFT

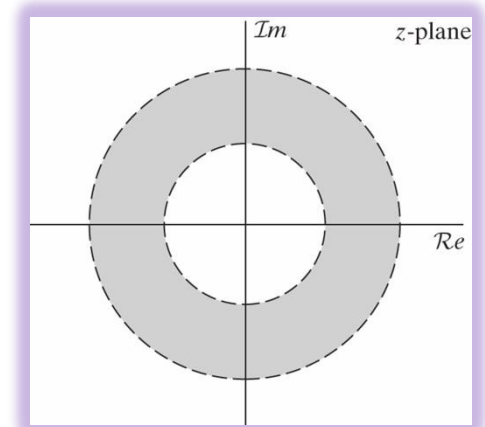
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- If $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

which is DTFT of $x[n]r^{-n}$.

- Convergence of DTFT
 - Is $x[n]$ absolutely summable?
- Convergence of z-Transform
 - Is $x[n]r^{-n}$ absolutely summable?
 - Therefore, the region of convergence will be a ring shape.



Why do we need the extension?

- Consider the DTFT pair

$$a^n u[n] \xleftrightarrow{F} \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1$$

What happens if $|a| \geq 1$?

- z-Transform pair

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

ROC
(region of convergence)

- z-Transform can be applied to a broader class of signals than DTFT
 - It is useful in studying a broader class of systems
 - It is used to analyze the causality and stability of a system

ROC should be specified

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

$$-a^n u[-n - 1] \xleftrightarrow{z} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

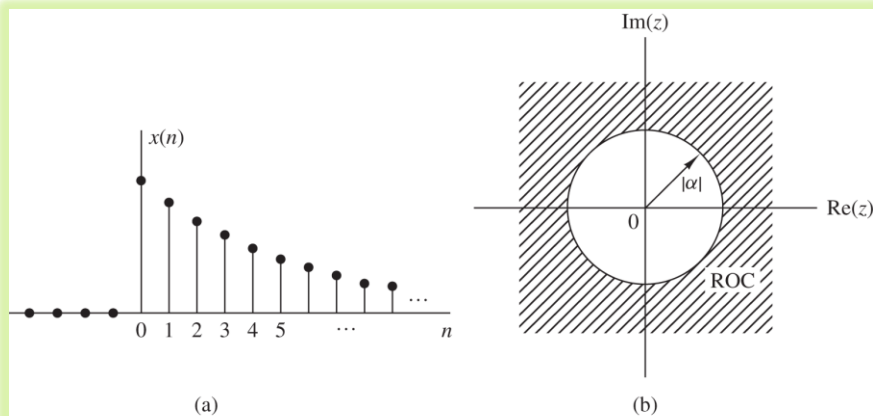


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z -transform (b).

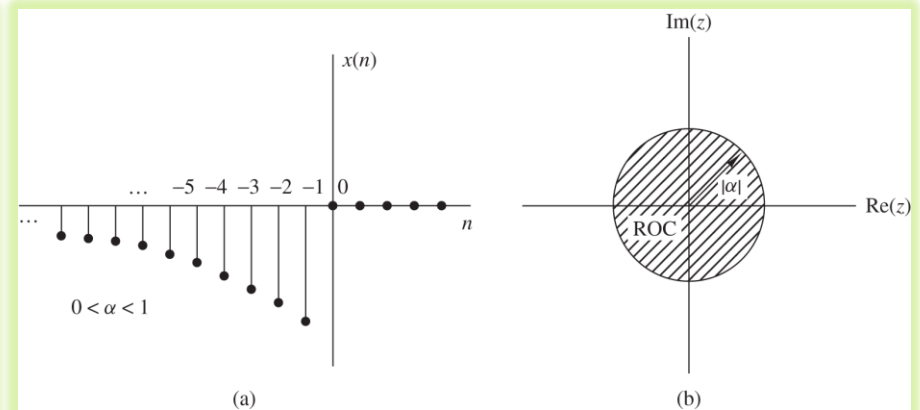
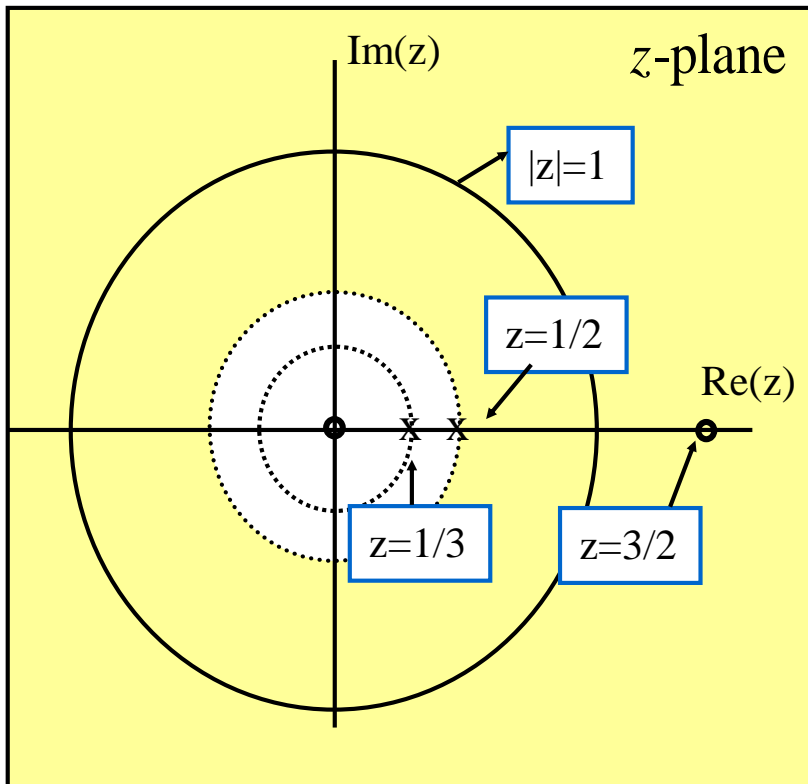


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n - 1)$ (a), and the ROC of its z -transform (b).

ROC should be specified

$$\text{Ex) } x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| > \frac{1}{2}$$

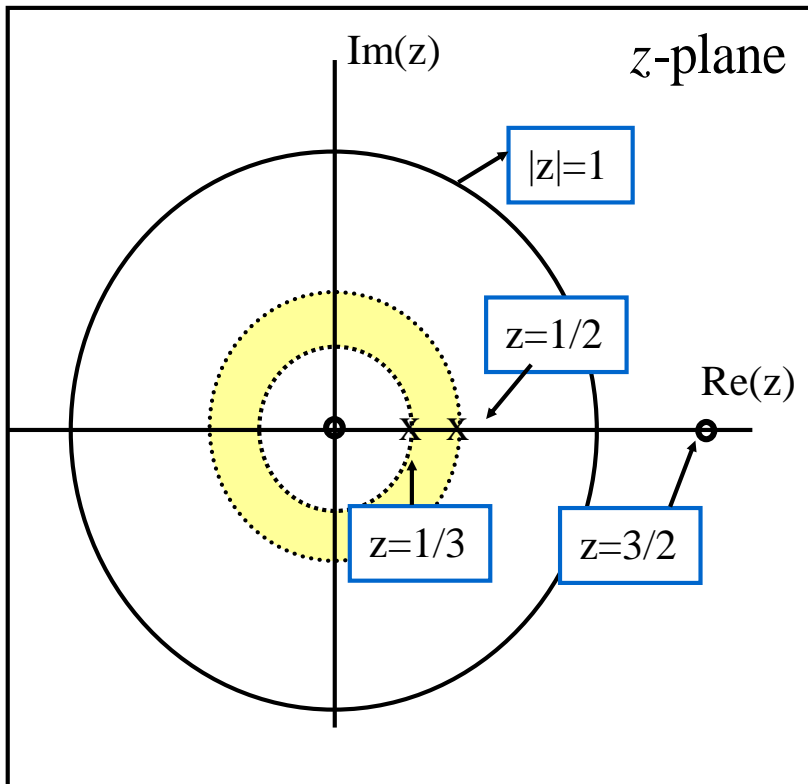


There are other sequences, which generate the same $X(z)$ but with different ROC's

ROC should be specified

Ex) $x[n] = ?$

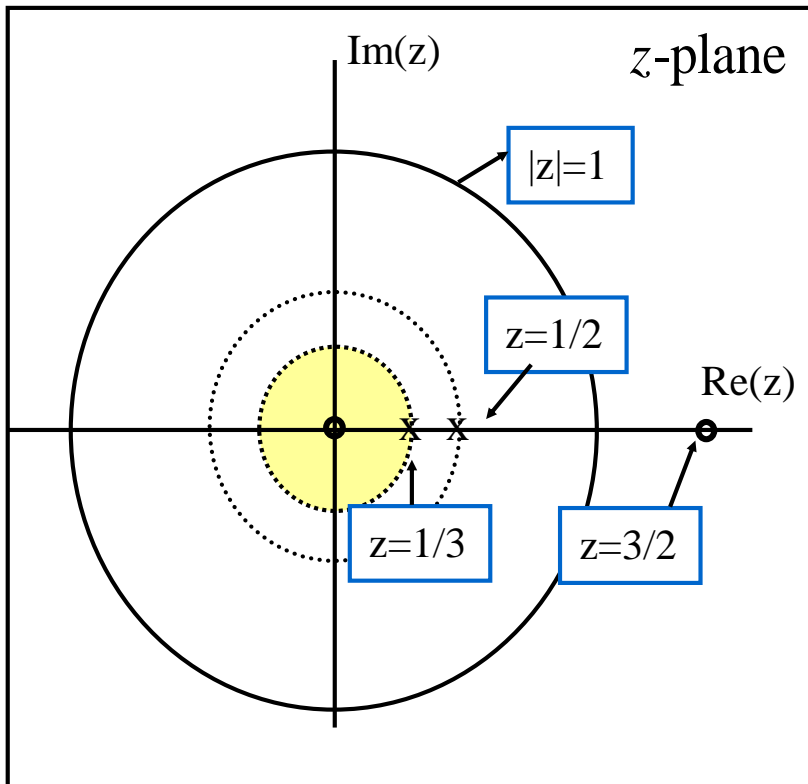
$$X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$



ROC should be specified

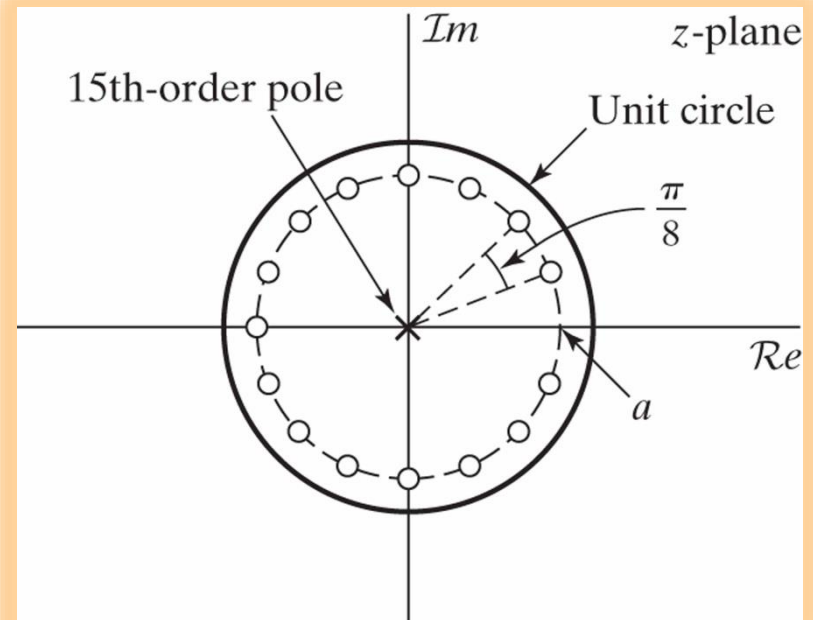
Ex) $x[n] = ?$

$$X(z) = \frac{z(z - \frac{3}{2})}{(z - \frac{1}{3})(z - \frac{1}{2})}, \quad |z| < \frac{1}{3}$$



Another Example

- $x[n] = \begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise.} \end{cases}$



Pole-zero plot when $N = 16$ and a is real such that $0 < a < 1$. The ROC in this example consists of all values of z except $z = 0$.

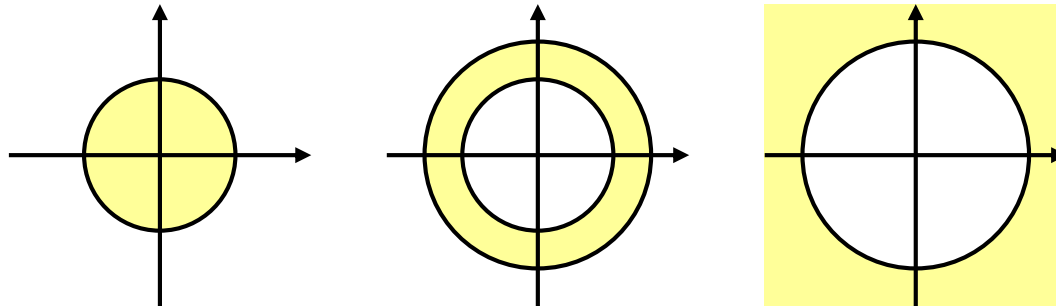
Common z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1 - z^{-1}}$	$ z > 1$
3. $-u[-n - 1]$	$\frac{1}{1 - z^{-1}}$	$ z < 1$
4. $\delta[n - m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$ z > a $
6. $-a^n u[-n - 1]$	$\frac{1}{1 - az^{-1}}$	$ z < a $
7. $na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z > a $
8. $-na^n u[-n - 1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z < a $
9. $\cos(\omega_0 n) u[n]$	$\frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
10. $\sin(\omega_0 n) u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$	$ z > 1$
11. $r^n \cos(\omega_0 n) u[n]$	$\frac{1 - r \cos(\omega_0)z^{-1}}{1 - 2r \cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z > r$
12. $r^n \sin(\omega_0 n) u[n]$	$\frac{r \sin(\omega_0)z^{-1}}{1 - 2r \cos(\omega_0)z^{-1} + r^2 z^{-2}}$	$ z > r$
13. $\begin{cases} a^n, & 0 \leq n \leq N - 1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$ z > 0$

Properties on ROC

- ROC of $X(z)$ consists of a single ring in the z -plane centered at the origin
 - Proof: Skipped. Refer to any textbook on complex analysis.

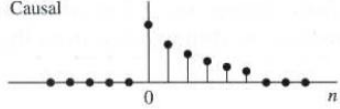
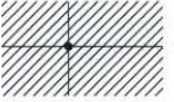
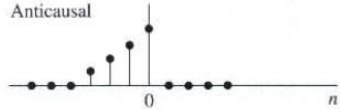
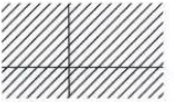
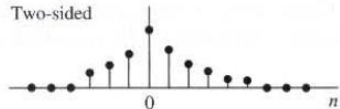
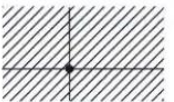
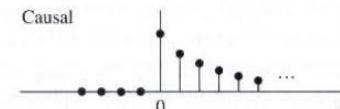


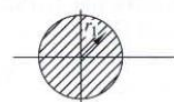
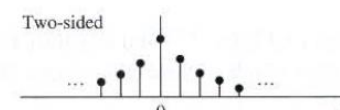



- ROC does not contain any poles.
- $x[n]$ has the Fourier transform, if ROC includes the unit circle

Properties on ROC

- Suppose that $X(z)$ is rational.
- $x[n]$ is a finite duration sequence
 \Rightarrow ROC is the entire z -plane
 except possibly $z = 0$ or $z = \infty$.
- $x[n]$ is right-sided
 \Rightarrow ROC is the region in the z -plane
 outside the outermost pole.
- $x[n]$ is left-sided
 \Rightarrow ROC is the region inside the
 innermost nonzero pole.
- $x[n]$ is two-sided
 \Rightarrow ROC is a ring, bounded on the
 interior and the exterior by poles.
- ROC is a connected region

TABLE 3.1 Characteristic Families of Signals with Their Corresponding ROCs

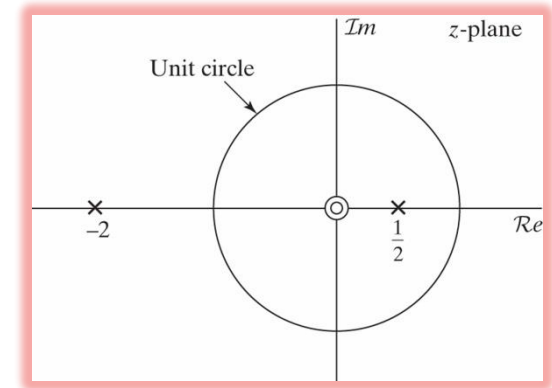
Signal	ROC
Finite-Duration Signals	
Causal 	 Entire z -plane except $z = 0$
Anticausal 	 Entire z -plane except $z = \infty$
Two-sided 	 Entire z -plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal 	 $ z > r_2$
Anticausal 	 $ z < r_1$
Two-sided 	 $r_2 < z < r_1$

Some signals don't have z-transforms

- $x[n] = \frac{1}{2^n} u[n] - \left(-\frac{1}{3}\right)^n u[-n - 1].$

Analysis of LTI Systems in z -Domain

- Causality
 - ROC of the system function is the **exterior of a circle**
- Stability
 - ROC contains the **unit circle**
- A causal system is stable if all poles are inside the unit circle



Inverse z-Transforms

Inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- C is any closed contour within the ROC of the z -transform
- Its proper evaluation requires some knowledge on complex integral
 - For example, you may refer to R. V. Churchill and J. W. Brown, *Complex Variables and Applications*, McGraw-Hill
- We do not use this formula. Instead, we decompose $X(z)$ into a number of terms, each of which can be inverse transformed using tables or partial fractions

Inverse z-Transform by Partial-Fraction Expansion

Ex 1) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Inverse z -Transform by Partial-Fraction Expansion

Ex 2) Determine the causal signal $x[n]$, whose z -transform is

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$$

Inverse z -Transform by Partial-Fraction Expansion

Ex 3) Determine the causal signal $x[n]$, whose z -transform is

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}$$

Inverse z-Transform by Partial-Fraction Expansion

Ex 4) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Inverse z -Transform by Power Series Expansion

Ex 5) Determine the signal $x[n]$, whose z -transform is

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|.$$

Properties

Properties of z -Transform

- Linearity

If $x_1[n] \Leftrightarrow X_1(z)$, ROC = R_{x_1}

$x_2[n] \Leftrightarrow X_2(z)$, ROC = R_{x_2}

Then

$$a x_1[n] + b x_2[n] \Leftrightarrow a X_1(z) + b X_2(z),$$

$$\text{ROC contains } R_{x_1} \cap R_{x_2}$$

Ex) Determine the z -transforms of
 $\cos(\omega_0 n)u[n]$ and $\sin(\omega_0 n)u[n]$

Ex) Determine the z -transform of
 $x[n] = a^n(u[n] - u[n - N])$.

Properties of z -Transform

- Time shifting

$$x[n - k] \Leftrightarrow z^{-k} X(z),$$

– ROC = R_x

(except for the possible addition or deletion of $z = 0$ or $z = \infty$)

Ex) Determine the inverse z -transform of

$$X(z) = \frac{1}{z - \frac{1}{4}}, \quad |z| > \frac{1}{4}$$

Properties of z -Transform

- Scaling in the z -domain

$$a^n x[n] \Leftrightarrow X(z/a), \quad \text{ROC} = |a|R_x.$$

Ex) Determine the z -transform of

$$x[n] = r^n \cos(\omega_0 n) u[n]$$

Properties of z -Transform

- Differentiation in the z -domain

$$nx[n] \Leftrightarrow -z \frac{dX(z)}{dz}, \quad \text{ROC} = R_x.$$

Ex) Determine the z -transform of $na^n u[n]$

Ex) Determine the signal $x[n]$ corresponding to

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

Properties of z-Transform

- Time reversal

$$x[-n] \Leftrightarrow X\left(\frac{1}{z}\right), \quad \text{ROC} = \frac{1}{R_x}.$$

Ex) Determine the z-transform of $a^{-n}u[-n]$

Properties of z -Transform

- Convolution becomes multiplication

$$x_1[n] * x_2[n] \Leftrightarrow X_1(z)X_2(z),$$

$$\text{ROC contains } R_{x_1} \cap R_{x_2}$$

Ex) Compute the convolution of

$$x_1[n] = \delta[n] + 2\delta[n - 1] + \delta[n - 2] \quad \text{and}$$

$$x_2[n] = \delta[n] - \delta[n - 1].$$

Properties of z-Transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0} X(z)$	R_x , except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$ z_0 R_x$
4	3.4.4	$nx[n]$	$-z \frac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

z-Transform and LTI Systems

System Function

- A system function is the z-transform of an impulse response

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} = \frac{Y(z)}{X(z)}$$

- If the system is given by CCDE

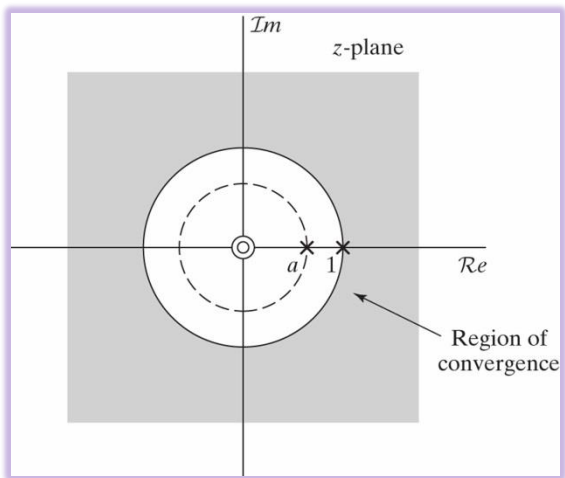
$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

then

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Example

- $h[n] = a^n u[n]$ and $x[n] = Au[n]$.



Example

- $y[n] = ay[n - 1] + x[n]$.
- We will see more applications of the z-transform in Chapter 5.

Analysis of LTI Systems in z -Domain

Ex) An LTI system is characterized by the system function

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}.$$

Specify the ROC of $H(z)$ and determine $h[n]$ for the following conditions

- a. The system is stable
- b. The system is causal

Unilateral z -Transform

Unilateral z -Transform and Its Properties

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- $y[n] = x[n - 1]$
 $\Leftrightarrow \mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z)$
- $y[n] = x[n - 2]$
 $\Leftrightarrow \mathcal{Y}(z) = x[-2] + x[-1]z^{-1} + z^{-2}\mathcal{X}(z)$
- and so forth

Example

$$y[n] - ay[n-1] = x[n] \text{ and } x[n] = u[n].$$

- Note that CCDE describes an LTI system only if we assume initial rest conditions.
- But, in this example, we assume $y[-1] \neq 0$.

$$y[n] = \begin{cases} y[-1], & n = -1, \\ y[-1]a^{n+1} + \frac{1}{1-a}(1 - a^{n+1}), & n \geq 0. \end{cases}$$

zero input response zero initial condition response