Digital Signal Processing

Chap 5. Transform Analysis of Linear Time-Invariant Systems

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LTI Systems

- Impulse response

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Frequency response

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)$$

  – Magnitude response

$$\left|Y\left(e^{j\omega}\right)\right| = \left|H\left(e^{j\omega}\right)\right| \cdot \left|X\left(e^{j\omega}\right)\right|$$

  – Phase response

$$\angle Y\left(e^{j\omega}\right) = \angle H\left(e^{j\omega}\right) + \angle X(e^{j\omega})$$

- System function

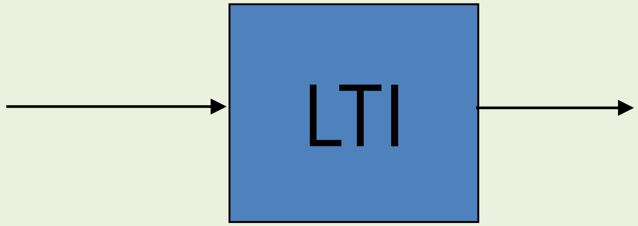
$$Y(z) = H(z)X(z)$$

Phase Response and Group Delay

Sinusoidal Input

- Assuming that $h[n]$ is real, we have the input-output relationship

$$A\cos(\omega_0 n + \phi)$$



$$A\,|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \theta(\omega_0))$$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$$

1. The amplitude is multiplied by $|H(e^{j\omega})|$
2. The output has a phase lag relative to the input by an amount $\theta(\omega) = \angle H(e^{j\omega})$

Principal Value of Phase

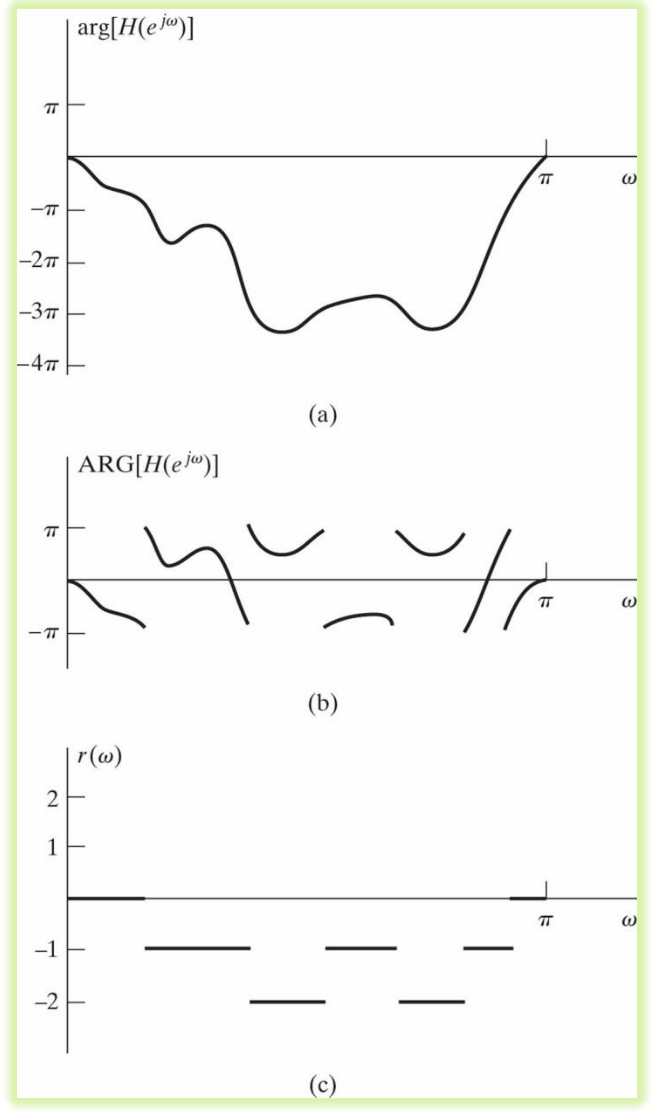
- Phase $\arg[H(e^{j\omega})] = \angle H(e^{j\omega})$ is not uniquely defined

- Principal value

$$-\pi < \text{ARG}[H(e^{j\omega})] \leq \pi$$

- $\arg[H(e^{j\omega})]$

$$= \text{ARG}[H(e^{j\omega})] + 2\pi r$$



Group Delay

- $\tau(\omega) = \mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\{\arg\left[H\left(e^{j\omega}\right)\right]\}$
- Ideal delay system
  - $h[n] = \delta[n - n_d] \;\Rightarrow\; \tau(\omega) = n_d$
- Linear-phase response is as good as zero-phase response in most applications
  - Ex) A lowpass filter with linear phase

$$H_{\mathrm{lp}}\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

- Group delay represents the linearity of phase

Group Delay

- Narrowband signal and group delay
  - Input: $x[n] = s[n]\cos(\omega_0 n)$
  - Linear approximation of phase:

$$\arg\left[H\left(e^{j\omega}\right)\right] \simeq -\phi_0 - \omega n_d$$

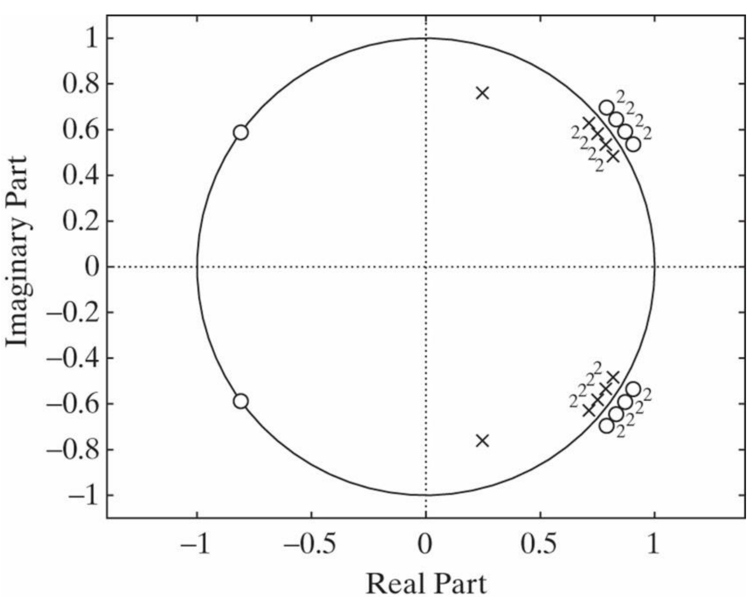
  - Output:

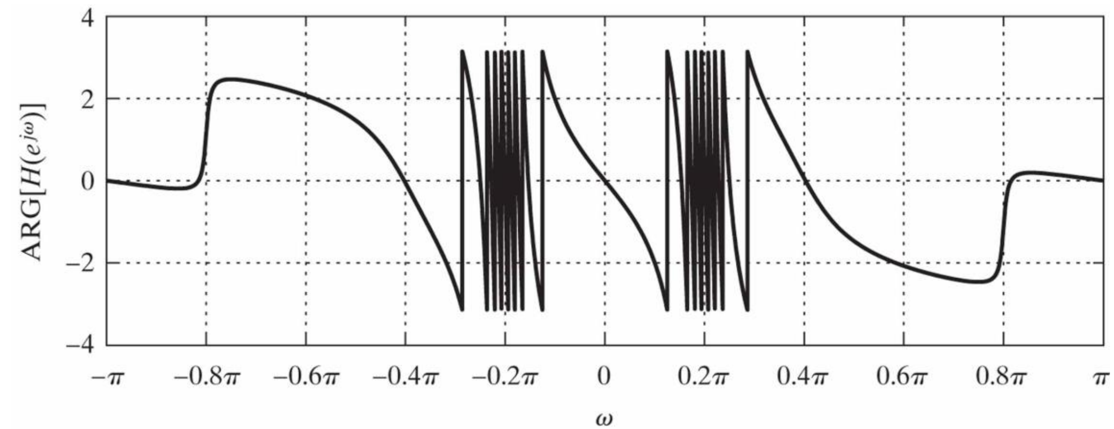
$$y[n] \simeq \left|H\left(e^{j\omega_0}\right)\right| s[n-n_d] \cos(\omega_0 n - \phi_0 - \omega_0 n_d)$$

- In other words, the time delay of the envelop of a narrowband signal around $\omega = \omega_0$ is given by the group delay $\tau(\omega_0)$.

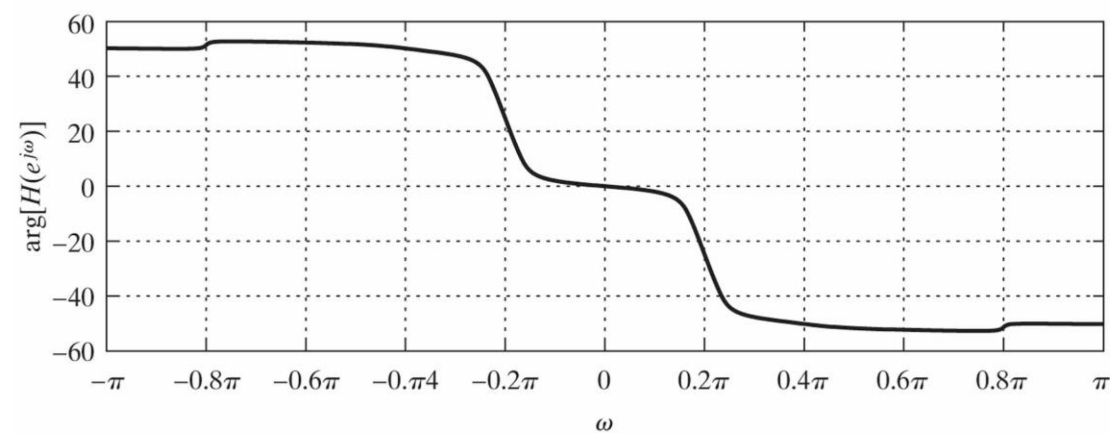
Group Delay: Example

$$H(z) = \frac{(1-0.98e^{j0.8\pi}z^{-1})(1-0.98e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.4\pi}z^{-1})(1-0.8e^{-j0.4\pi}z^{-1})}\prod_{k=1}^{4}\left(\frac{(c_k^*-z^{-1})(c_k-z^{-1})}{(1-c_k z^{-1})(1-c_k^* z^{-1})}\right)^2$$





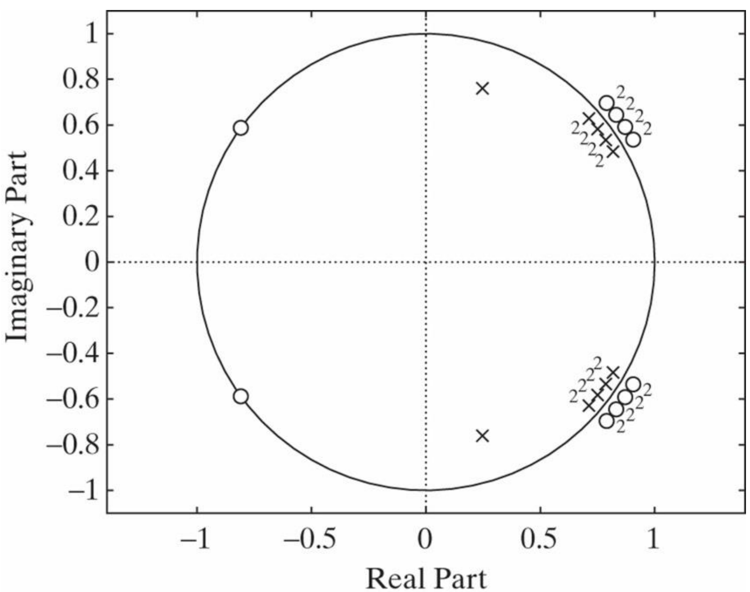
(a) Principle Value of Phase Response

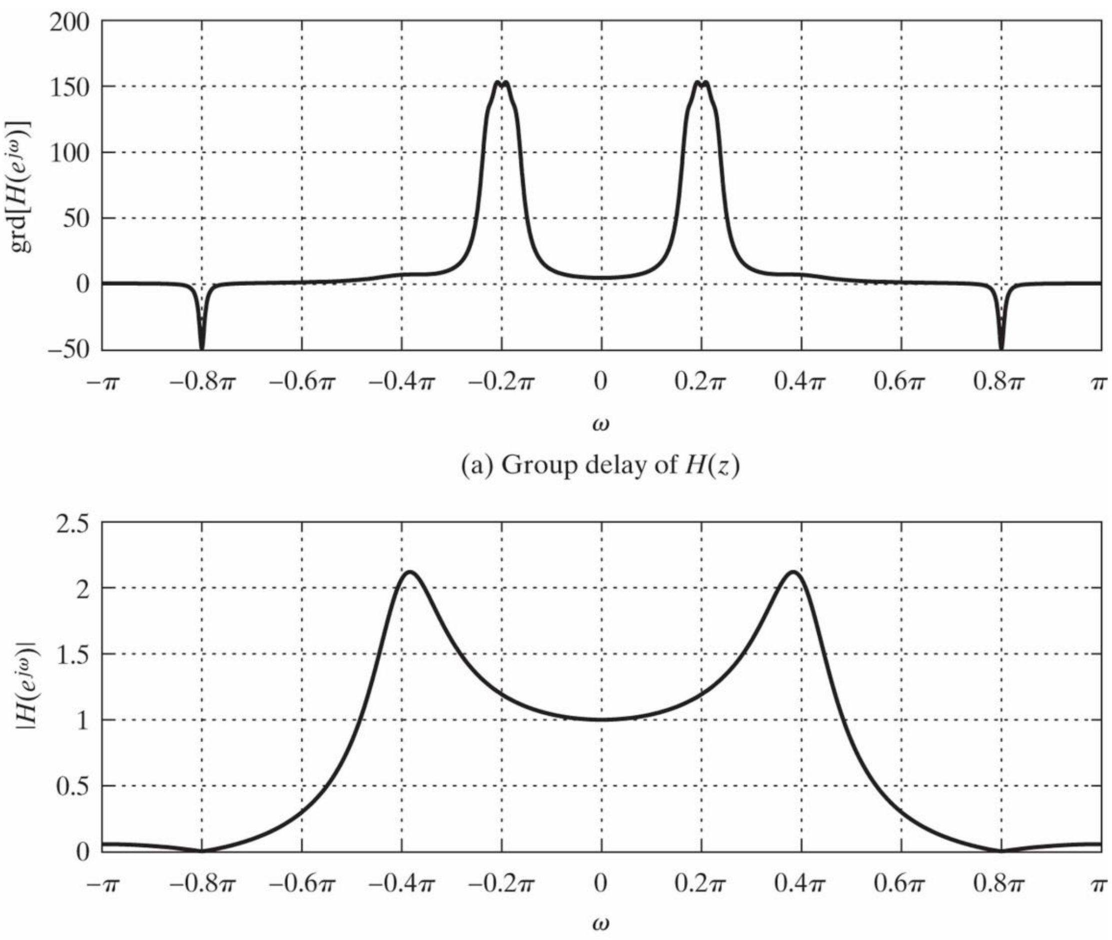


(b) Unwrapped Phase Response

Group Delay: Example

$$H(z) = \frac{(1-0.98e^{j0.8\pi}z^{-1})(1-0.98e^{-j0.8\pi}z^{-1})}{(1-0.8e^{j0.4\pi}z^{-1})(1-0.8e^{-j0.4\pi}z^{-1})}\prod_{k=1}^{4}\left(\frac{(c_k^*-z^{-1})(c_k-z^{-1})}{(1-c_kz^{-1})(1-c_k^*z^{-1})}\right)^2$$



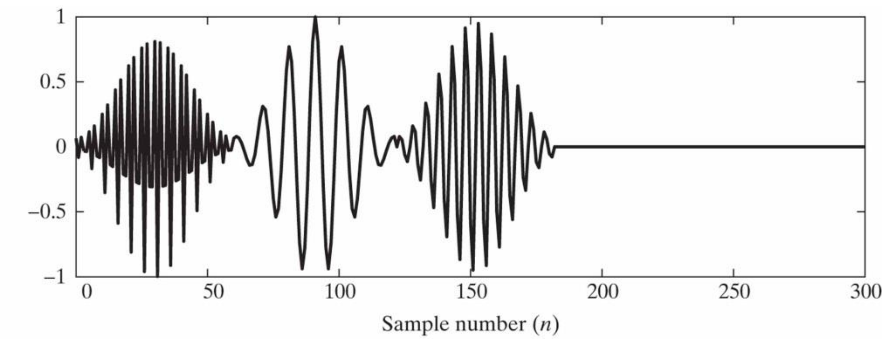


(a) Group delay of $H(z)$

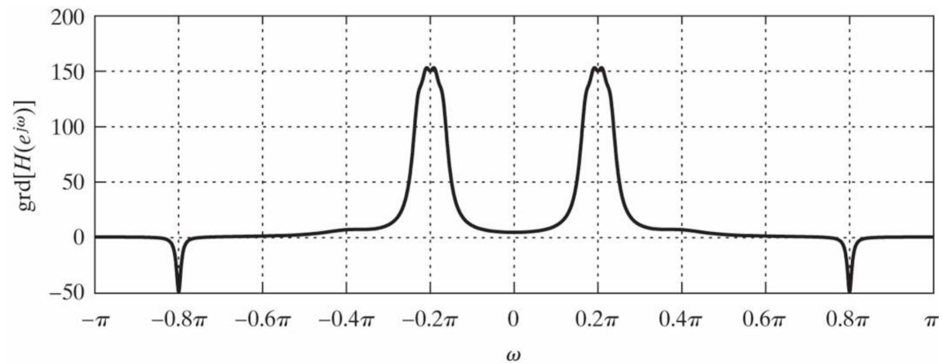
(b) Magnitude of Frequency Response

Group Delay: Example

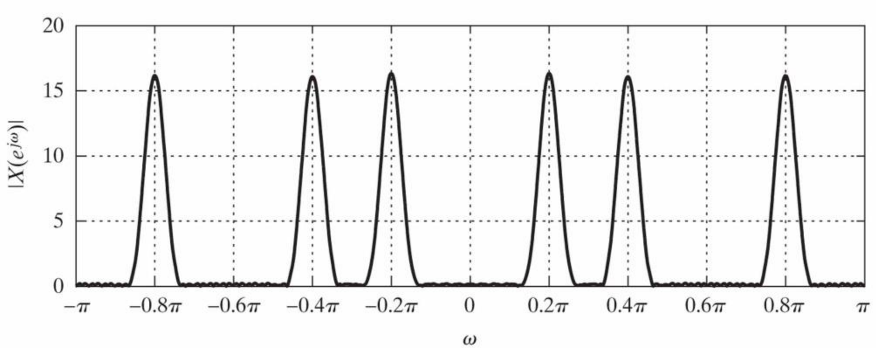
- $x[n] = x_3[n] + x_1[n-61] + x_2[n-122]$
- $x_1[n] = w[n]\cos(0.2\pi n),\ x_2[n] = w[n]\cos\left(0.4\pi n - \frac{\pi}{2}\right), x_3[n] = w[n]\cos\left(0.8\pi n + \frac{\pi}{5}\right)$
- $w[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{60}\right),\ 0 \le \mathrm{n} \le 60$



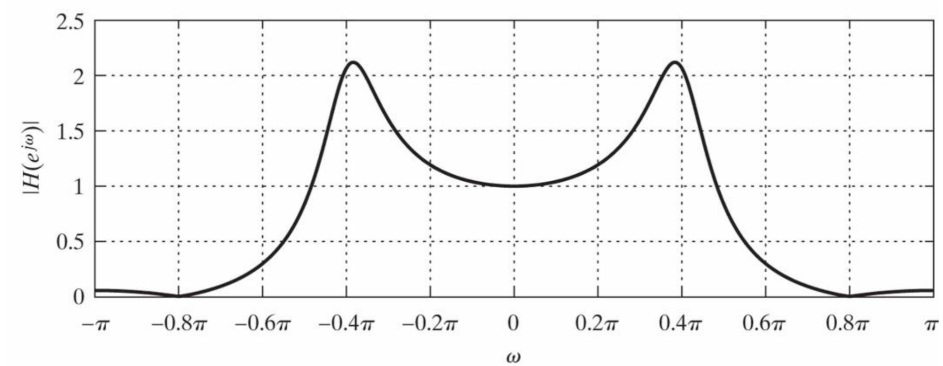
(a) Waveform of signal $x[n]$



(a) Group delay of $H(z)$



(b) Magnitude of DTFT of $x[n]$



(b) Magnitude of Frequency Response

Group Delay: Example

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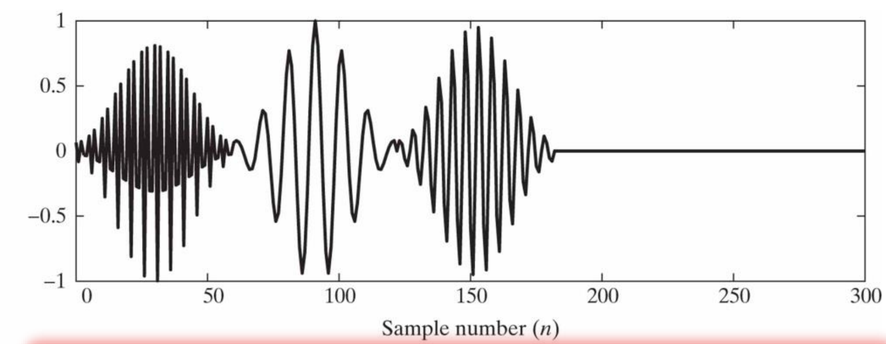
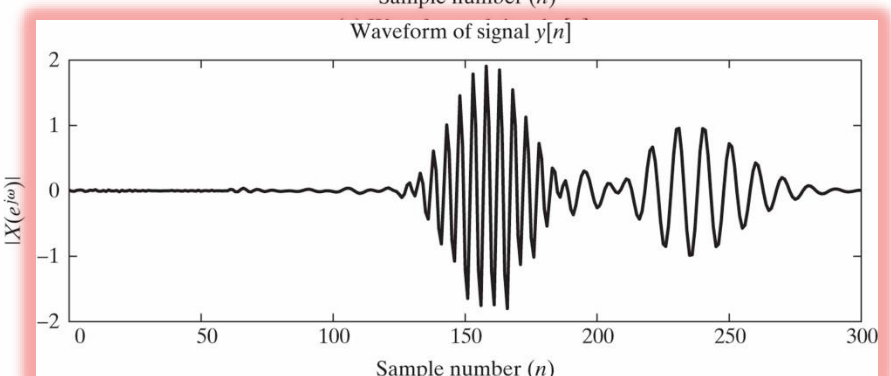
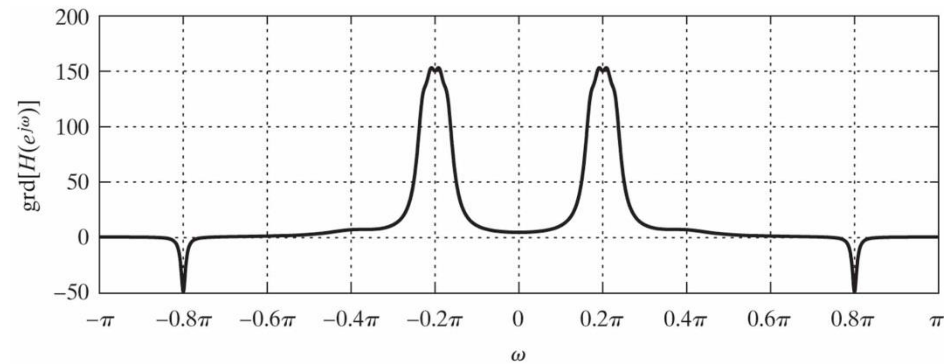


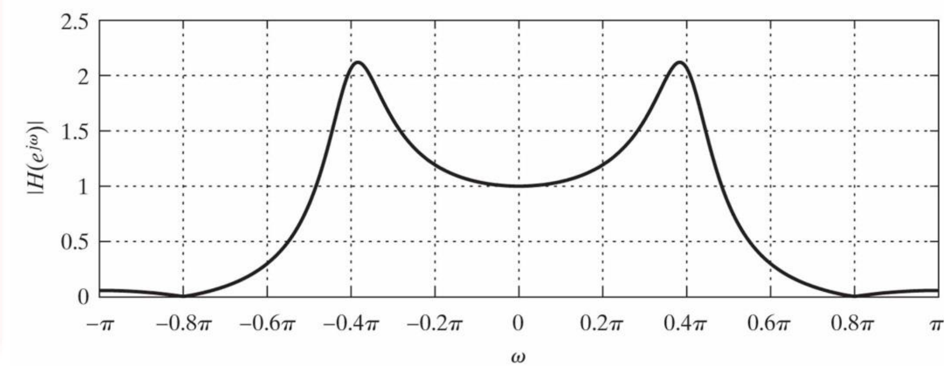
Waveform of signal $y[n]$



(b) Magnitude of DTFT of $x[n]$

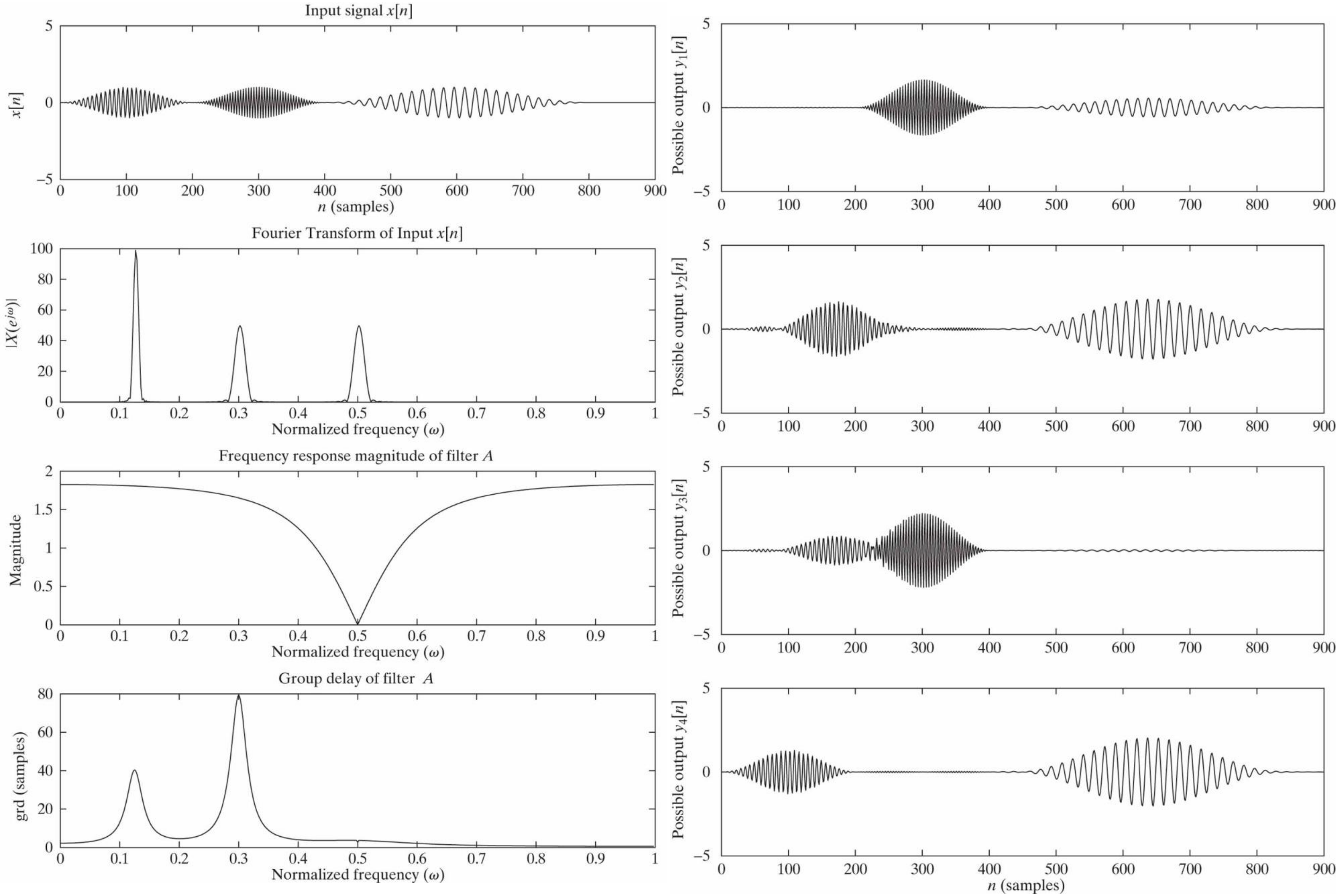


(a) Group delay of $H(z)$



(b) Magnitude of Frequency Response

Another Example



Systems Implemented by CCDE's

Constant-Coefficient Difference Equations

- CCDE

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

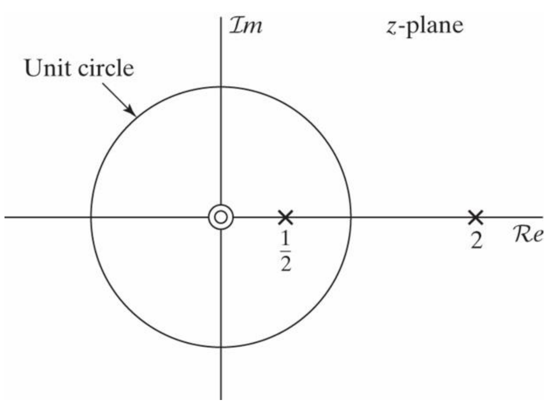
- (Rational) System Function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- Ex) $H(z) = \frac{\left(1+z^{-1}\right)^2}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{3}{4}z^{-1}\right)}$. Corresponding CCDE?

CCDE: Stability and Causality

- Stability: ROC contains the unit circle
- Causality: ROC is the outside of the outermost pole
- All poles of a causal stable system are inside the unit circle
- Ex) $y[n] - \frac{5}{2} y[n-1] + y[n-2] = x[n]$

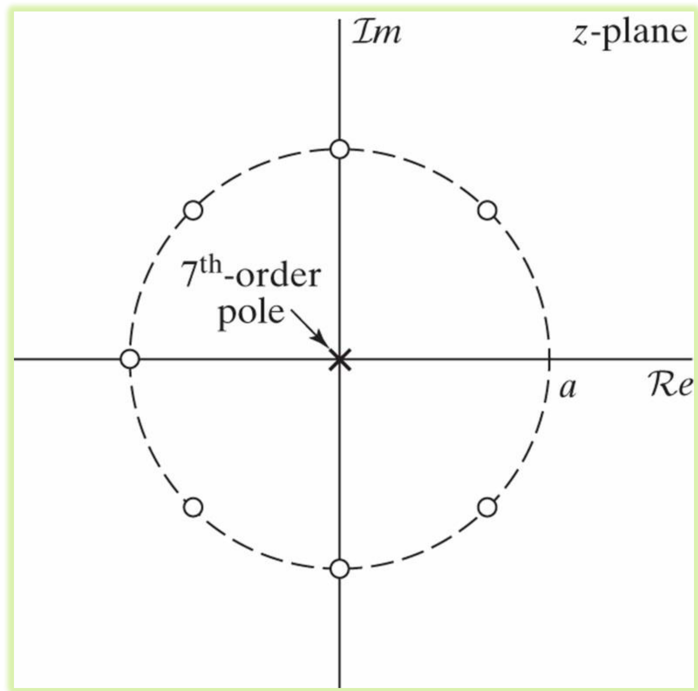


CCDE: Inverse Systems

- $H(z)H_i(z) = 1$ or $h[n] * h_i[n] = \delta[n]$
  - The ROC of $H_i(z)$ must overlap with that of $H(z)$
- $H(z) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \Rightarrow H_i(z) = \frac{a_0}{b_0}\frac{\prod_{k=1}^{N}(1-d_k z^{-1})}{\prod_{k=1}^{M}(1-c_k z^{-1})}$
- An LTI system is stable and causal and also has a stable and causal inverse if and only if both poles and zeros of $H(z)$ are inside the unit circle (minimum-phase system)
- Ex1) $H(z) = \frac{1-0.5z^{-1}}{1-0.9z^{-1}},\ |z| > 0.9$
- Ex2) $H(z) = \frac{z^{-1}-0.5}{1-0.9z^{-1}},\ |z| > 0.9$

CCDE: Impulse Responses

- IIR system: At least one nonzero pole is not canceled by a zero
  - Ex) $G(z) = \frac{1}{1-az^{-1}},\ |z| > |a|$
- FIR system: $H(z)$ has no poles except at $z = 0$.
  - Ex) $H(z) = \frac{1-a^{M+1}z^{-M-1}}{1-az^{-1}}$



Frequency Responses for Rational System Functions

$$H(e^{j\omega}) = \frac{b_0}{a_0}\frac{\prod_{k=1}^{M}(1 - c_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k e^{-j\omega})}$$

- Magnitude $|H(e^{j\omega})| =$

$$\left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}|1 - c_k e^{-j\omega}|}{\prod_{k=1}^{N}|1 - d_k e^{-j\omega}|}$$

- Gain (dB) =

$$20\log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20\log_{10}|1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20\log_{10}|1 - d_k e^{-j\omega}|$$

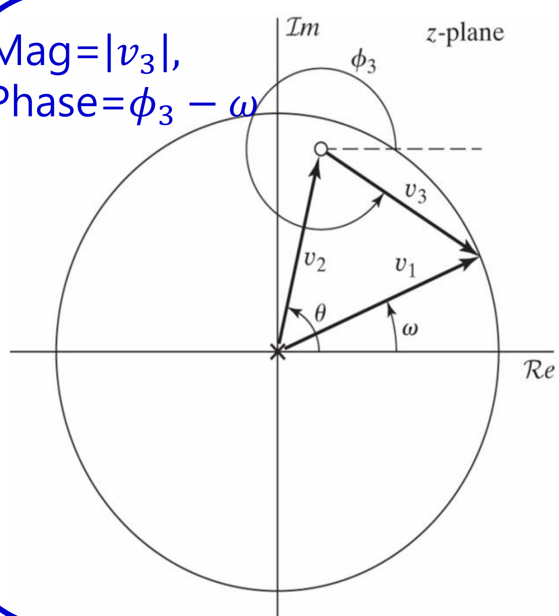
- Phase $\arg[H(e^{j\omega})] =$

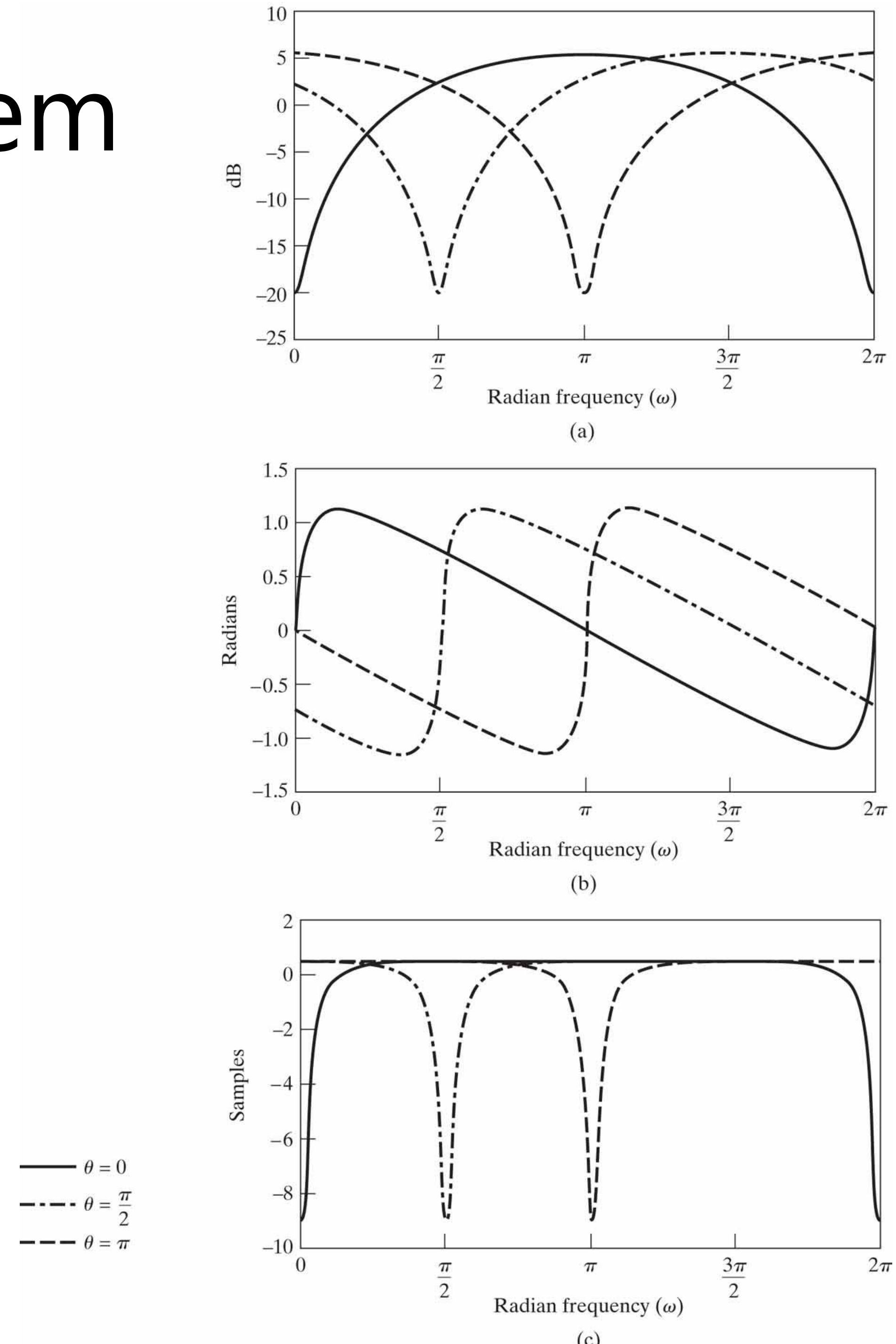
$$\arg[\frac{b_0}{a_0}] + \sum_{k=1}^{M}\arg[(1 - c_k e^{-j\omega})] - \sum_{k=1}^{N}\arg[(1 - d_k e^{-j\omega})]$$

1st-Order System

- $\left(1 - re^{j\theta}e^{-j\omega}\right)$
- Gain:
  $10\log_{10}(1 + r^2 - 2r\cos(\omega - \theta))$
- Phase: $\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$
- Group delay: $\frac{r^2 - r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$

Mag=$|v_3|$,
Phase=$\phi_3 - \omega$



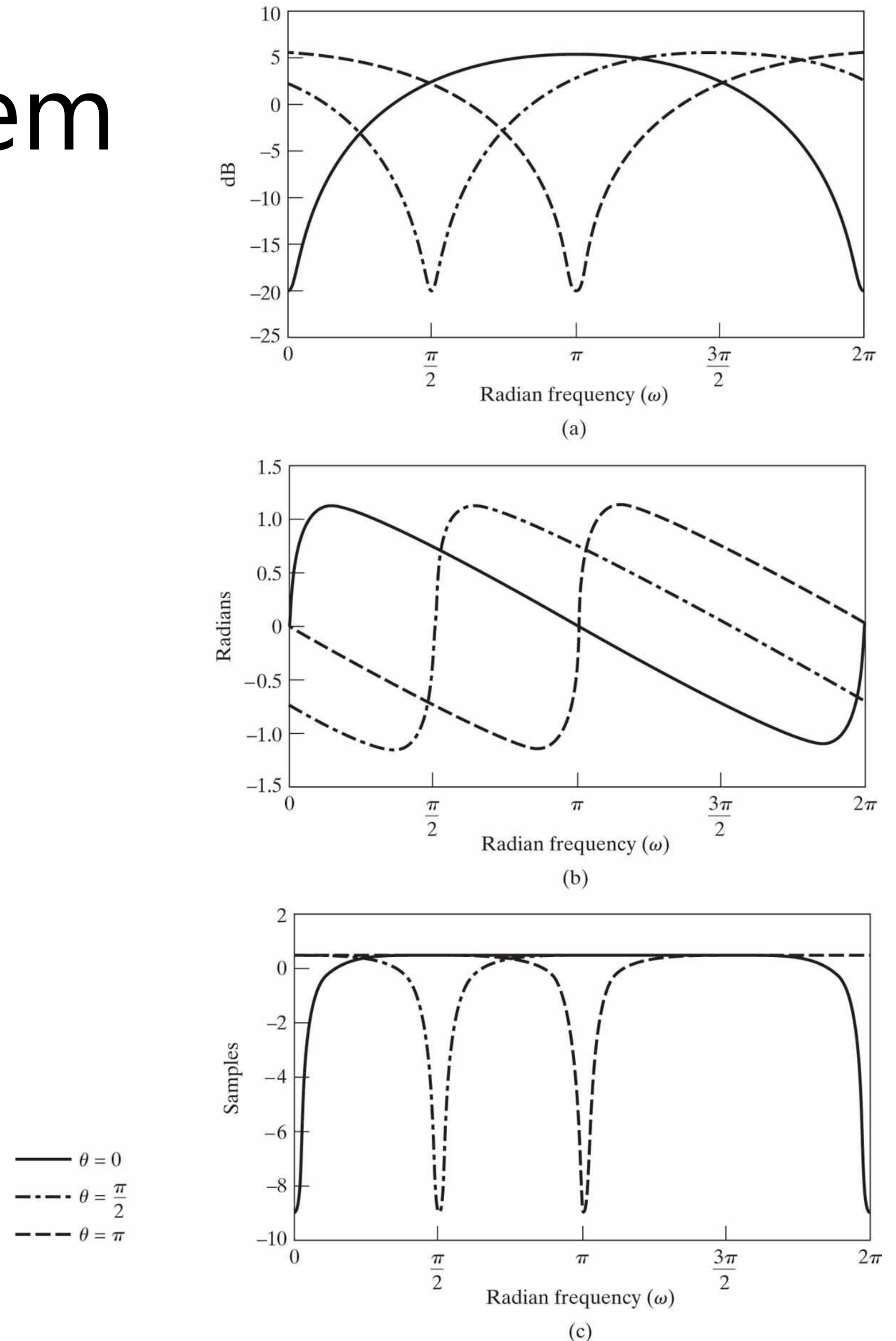


1st-Order System

- $\left(1 - re^{j\theta}e^{-j\omega}\right)$
- Gain: $10\log_{10}(1 + r^2 - 2r\cos(\omega - \theta))$
- Phase: $\arctan\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$
- Group delay: $\frac{r^2-r\cos(\omega-\theta)}{1+r^2-2r\cos(\omega-\theta)}$

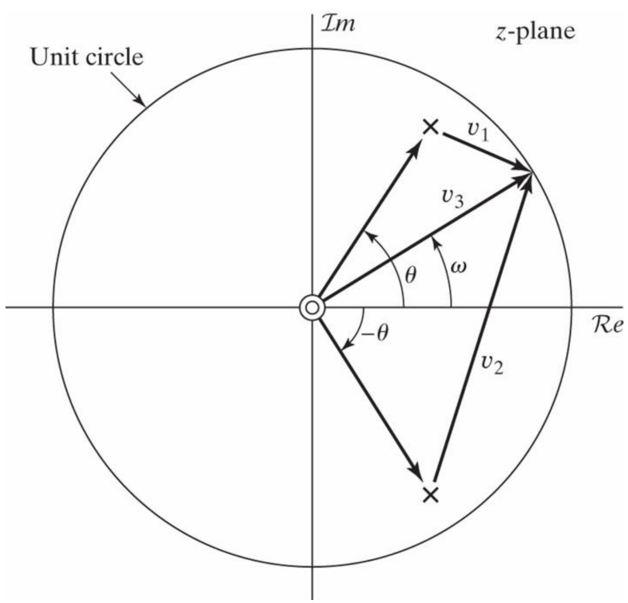
- Smaller magnitude and negative group delay near a zero
- cf) Bigger magnitude and positive group delay near a pole

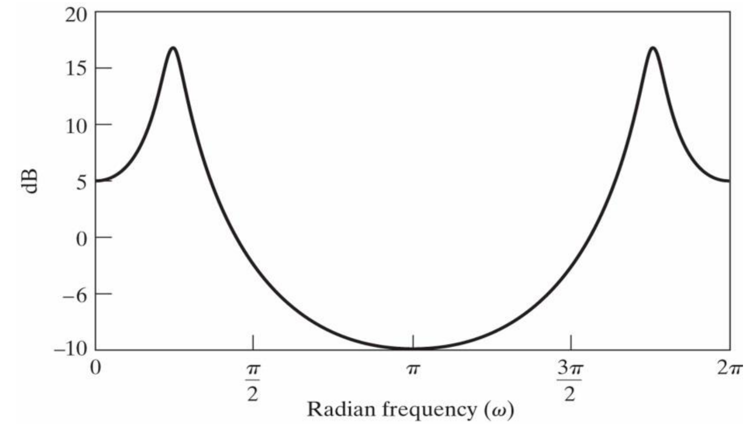
$$H\left(e^{j\omega}\right) = 1/\left(1 - re^{j\theta}e^{-j\omega}\right)$$



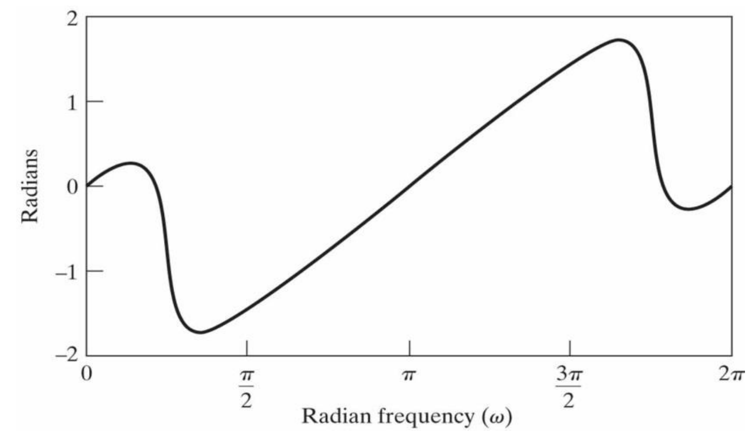
2^{nd}-Order IIR System

- $H(z) = \frac{1}{(1-r^{j\theta}z^{-1})(1-r^{-j\theta}z^{-1})}$
- $h[n] = \frac{r^n \sin[\theta(n+1)]}{\sin\theta} u[n]$





(a)



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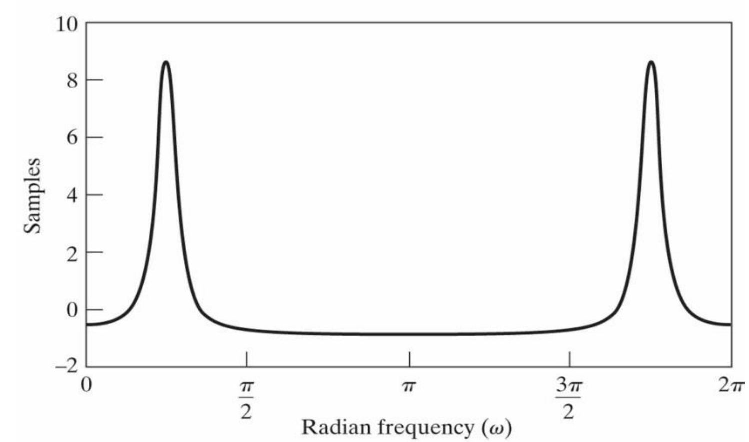
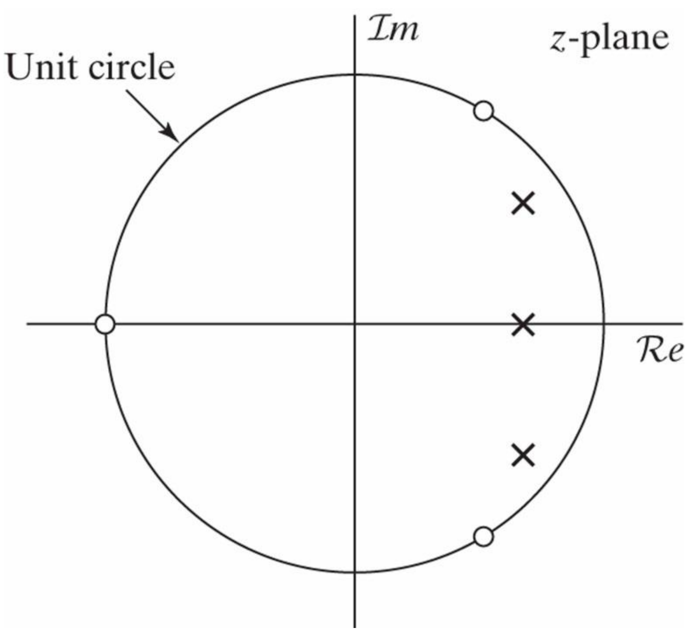
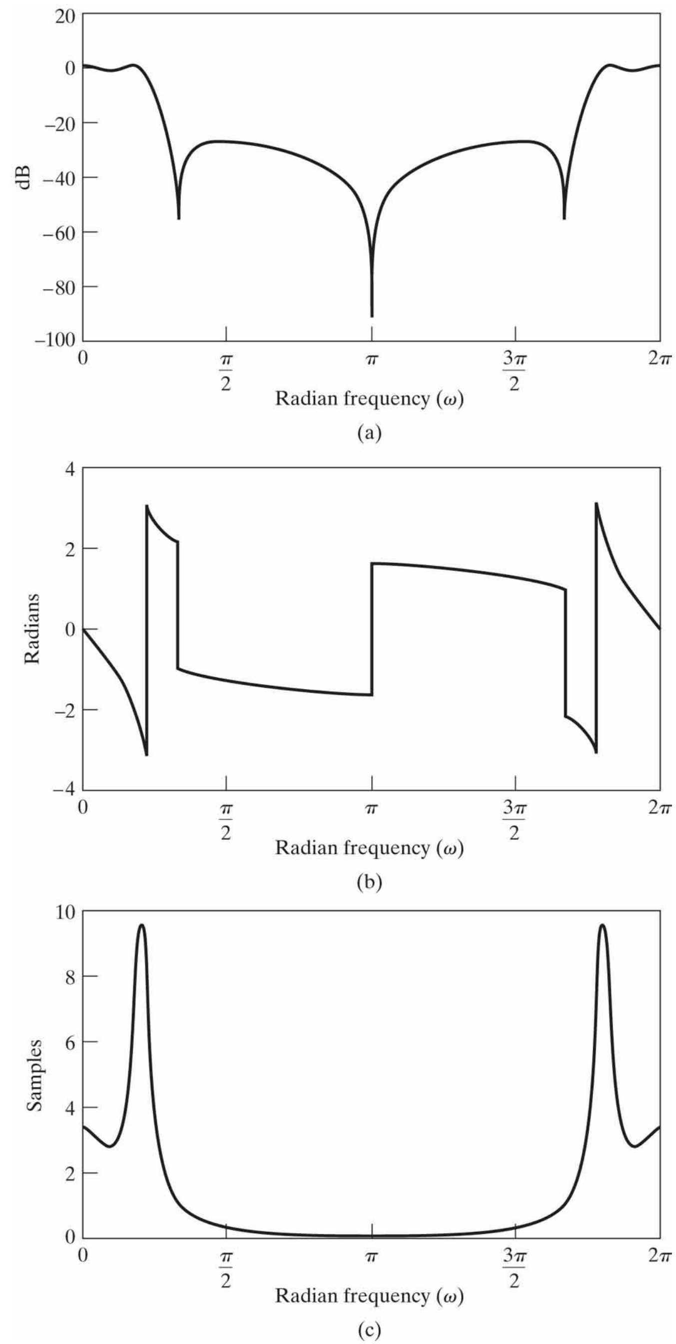


(c)

3rd-Order IIR System

- $H(z) = \dfrac{0.056(1+z^{-1})(1-1.017z^{-1}+z^{-2})}{(1-0.683z^{-1})(1-1.446z^{-1}+0.796z^{-2})}$





Allpass Systems and Minimum-Phase Systems

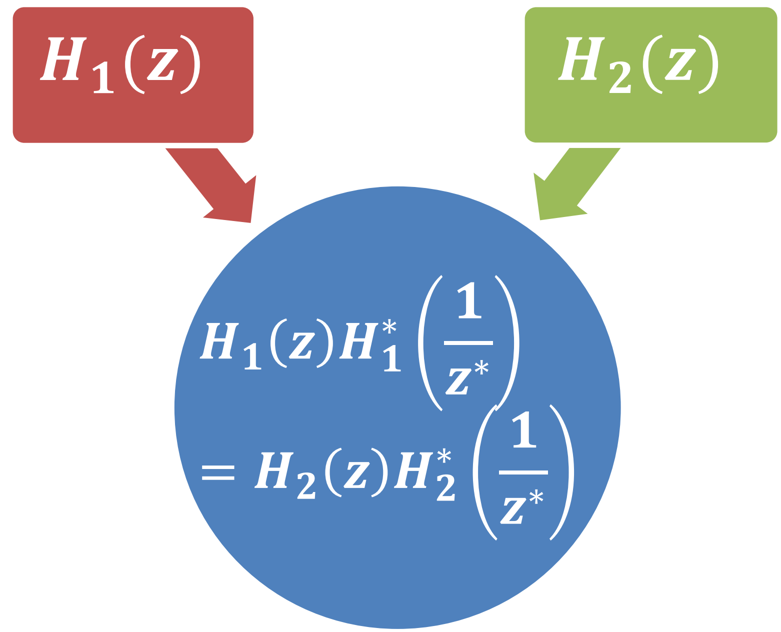
Different Systems with the Same Magnitude Response

- From now on, we focus on rational system functions, which can be implemented by CCDE's
- (Let's accept this without proof) If $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$, then

$$H_1(z)H_1^*\left(\frac{1}{z^*}\right) = H_2(z)H_2^*\left(\frac{1}{z^*}\right)$$

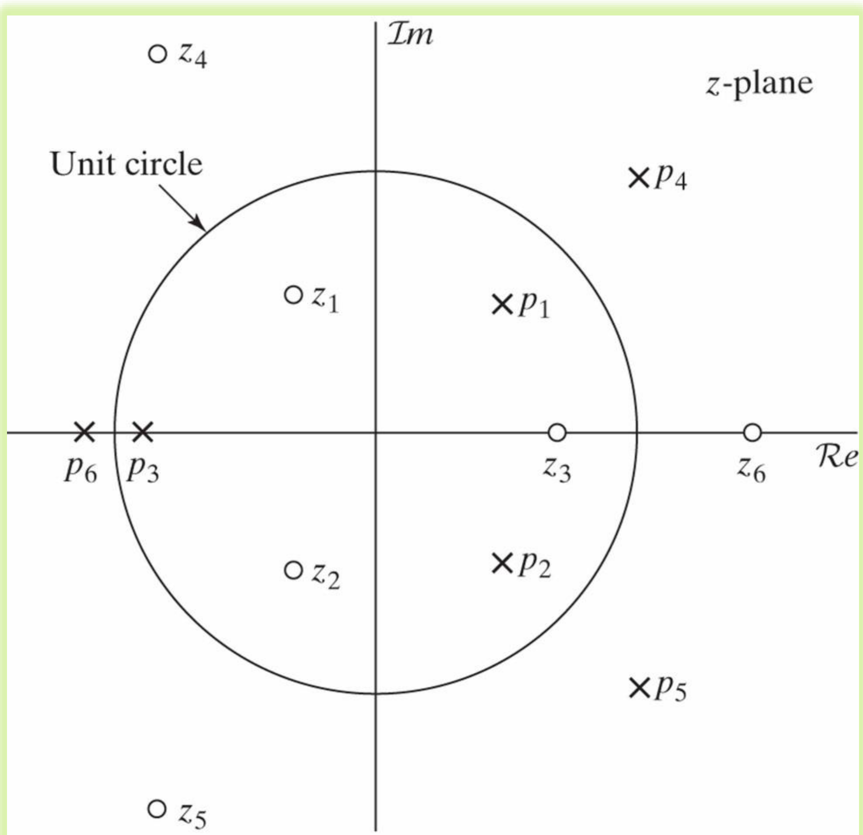
- Note that

$$H_1(z)H_1^*\left(\frac{1}{z^*}\right)\Big|_{z=e^{j\omega}} = \left|H_{1(e^{j\omega})}\right|^2$$



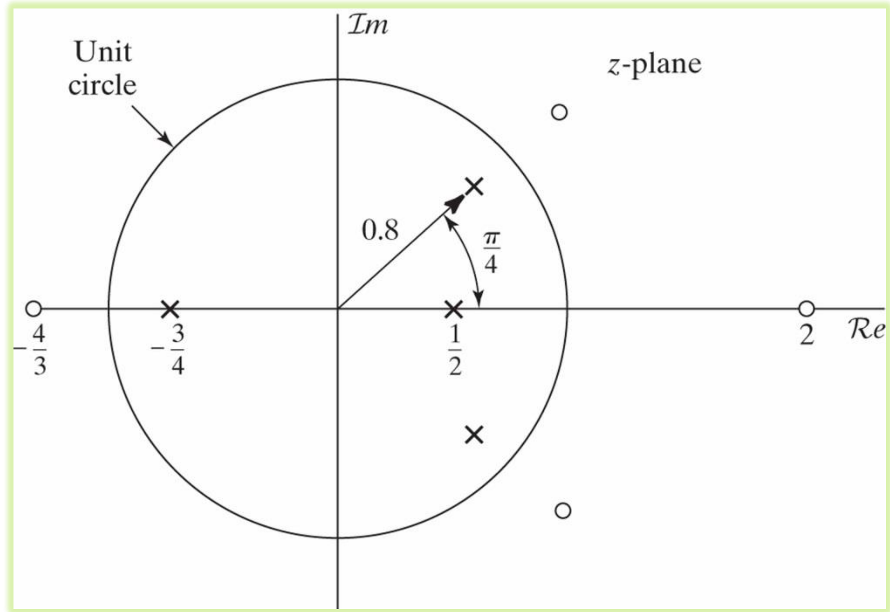
Different Systems with the Same Magnitude Response

- $H(z)H^*\left(\frac{1}{z^*}\right)$ is given below. What is $H(z)$?



Allpass Systems

- An allpass system has unity magnitude $|H_{\mathrm{ap}}(e^{j\omega})| = 1$ for all ω
- $H_{\mathrm{ap}}(z) = \frac{z^{-1} - a^*}{1 - a z^{-1}}$



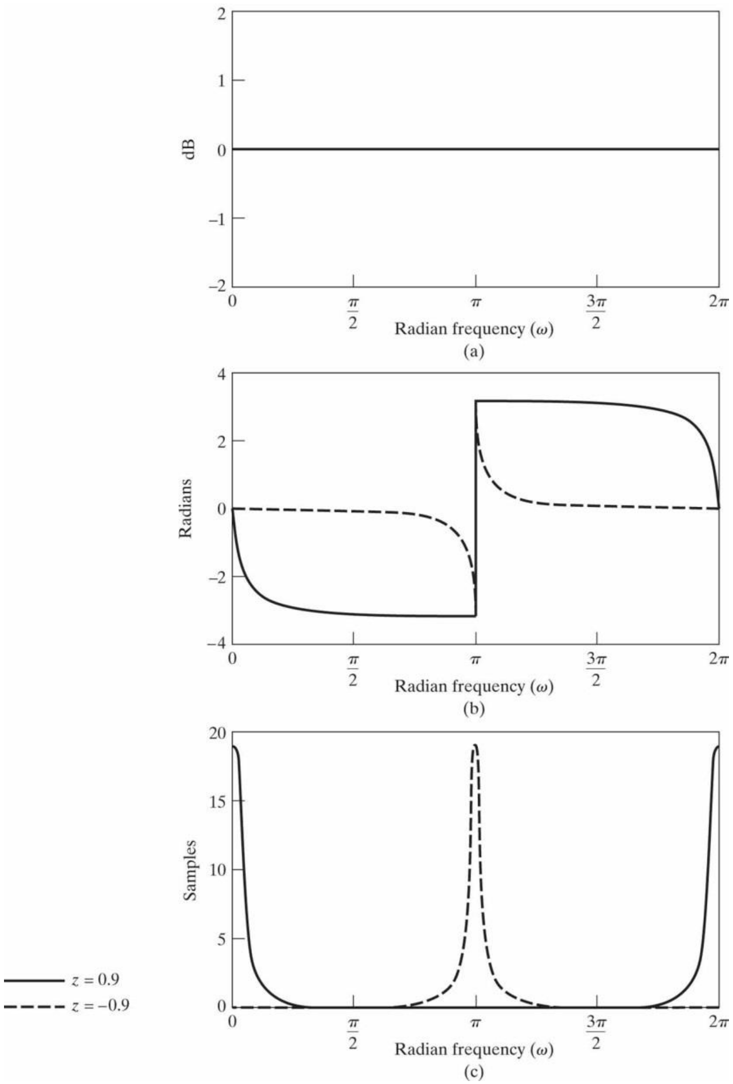
- In general, for a real-valued impulse response

$$H_{\mathrm{ap}}(z) = \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

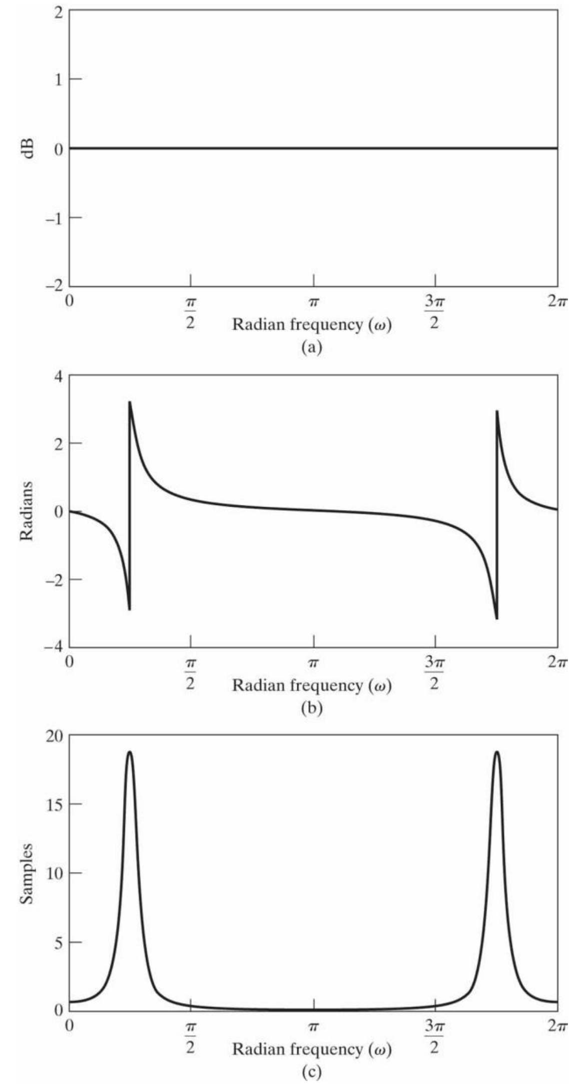
Allpass Systems

- $H_{\text{ap}}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}} \Rightarrow H_{\text{ap}}\left(e^{j\omega}\right) = \frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}$
- $\text{angle}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = -\omega - 2\arctan\left[\frac{r\sin(\omega - \theta)}{1 - r\cos(\omega - \theta)}\right]$
- $\text{grd}\left[\frac{e^{-j\omega} - re^{-j\theta}}{1 - re^{j\theta}e^{-j\omega}}\right] = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \theta)}$
- The group delay of a causal, stable allpass system is always positive

Allpass Systems



One pole at $z = 0.9$ or -0.9



Two poles at $z = 0.9e^{\frac{j\pi}{4}}$ and $0.9e^{-\frac{j\pi}{4}}$

Minimum-Phase Systems

- A minimum-phase system is a system with all poles and zeros inside the unit circle
- Any rational system function can be decomposed into

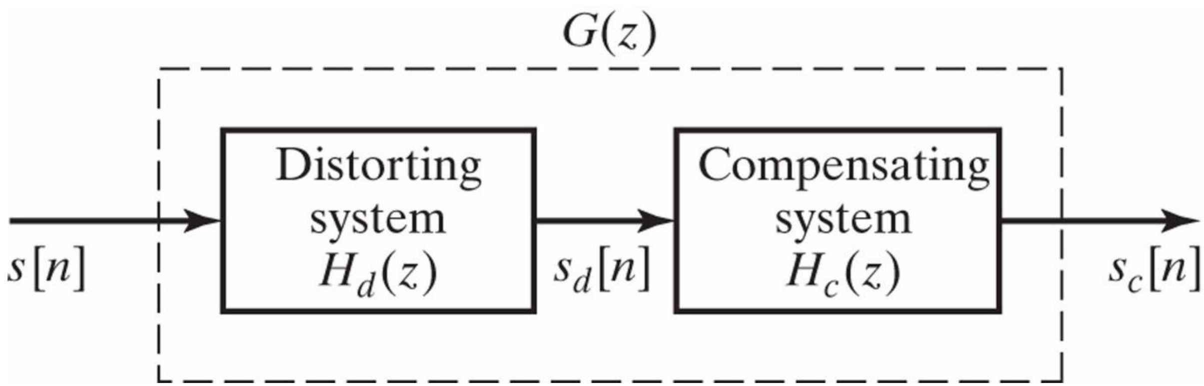
$$H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$$

- Ex1) $H_1(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}}$

- Ex2) $H_2(z) = \frac{(1+\frac{3}{2}e^{j\frac{\pi}{4}}z^{-1})(1+\frac{3}{2}e^{-j\frac{\pi}{4}}z^{-1})}{1-\frac{1}{3}z^{-1}}$

Minimum-Phase Systems

- Distortion compensation

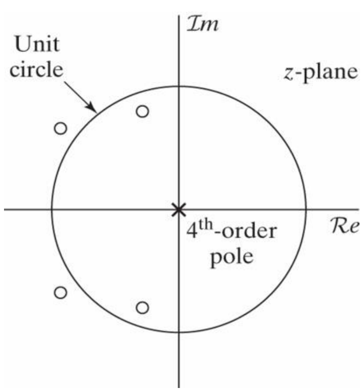


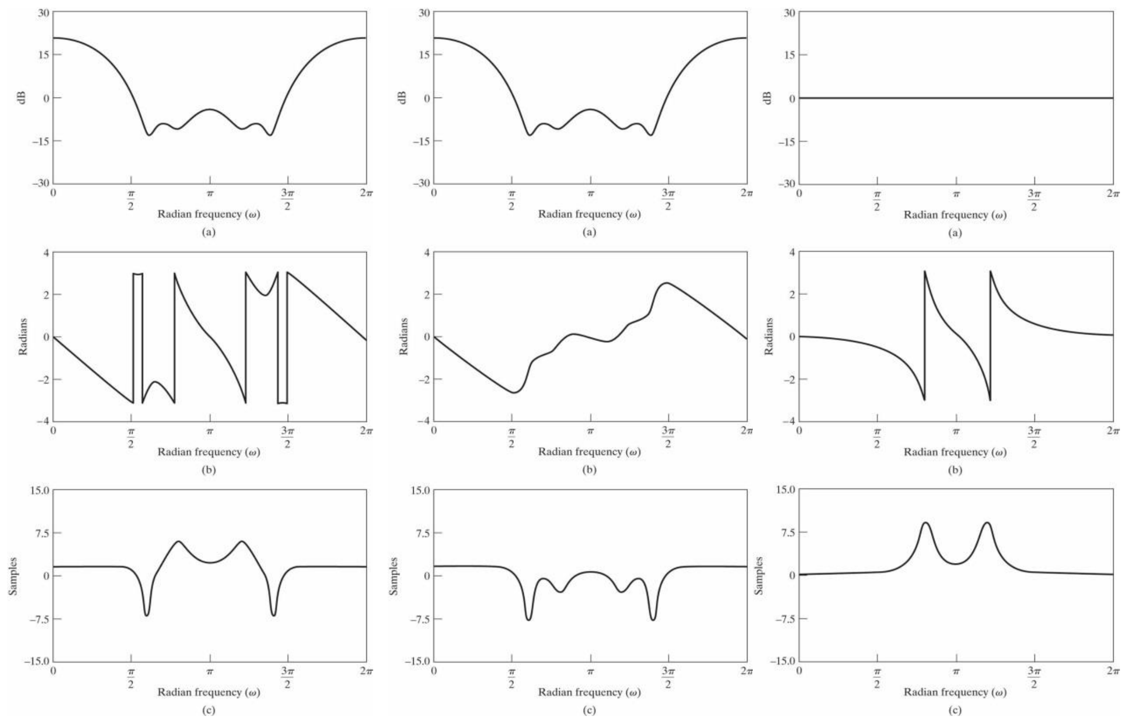
  - $H_d(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$
  - $H_c(z) = 1/H_{\text{min}}(z)$
  - $G(z) = H_{\text{ap}}(z)$

Minimum-Phase Systems

- Ex) $$H_d(z) = \frac{\left(1-0.9e^{j0.6\pi}z^{-1}\right)\left(1-0.9e^{-j0.6\pi}z^{-1}\right)}{\left(1-1.25e^{j0.8\pi}z^{-1}\right)\left(1-1.25e^{j0.8\pi}z^{-1}\right)}$$

$\boldsymbol{H_d(z) \quad = \quad H_{\min}(z) \quad \times \quad H_{ap}(z)}$





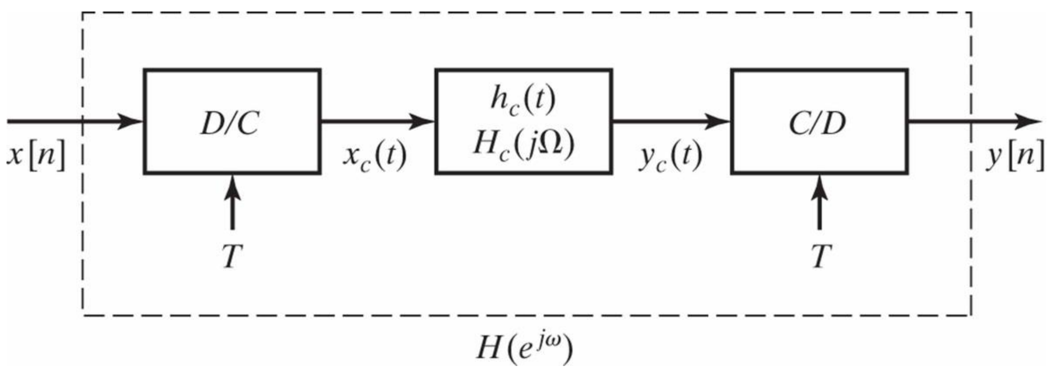
Minimum-Phase Systems

- Among causal, stable systems with the same magnitude response, **the minimum-phase system minimizes the group delay** because a causal, stable allpass system has a positive group delay

Linear-Phase Systems

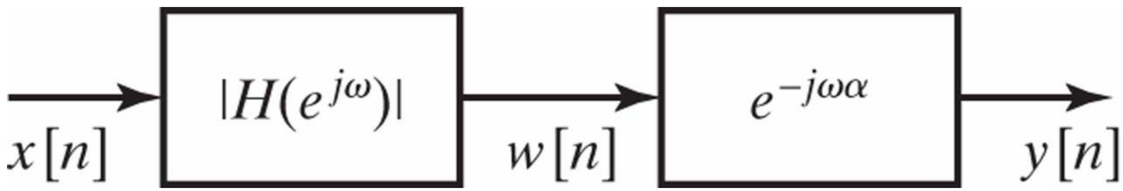
Review of Ideal Delay

- $H_{\mathrm{id}}(e^{j\omega}) = e^{-j\omega\alpha}, \quad h_{id}[n] = \dfrac{\sin \pi(n-\alpha)}{\pi(n-\alpha)}$



Linear-Phase Systems

- $H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$



- Ex) $H_{\mathrm{lp}}(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$

  What is $h_{\mathrm{lp}}[n]$?

Generalized Linear-Phase Systems

- $H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega\alpha}$

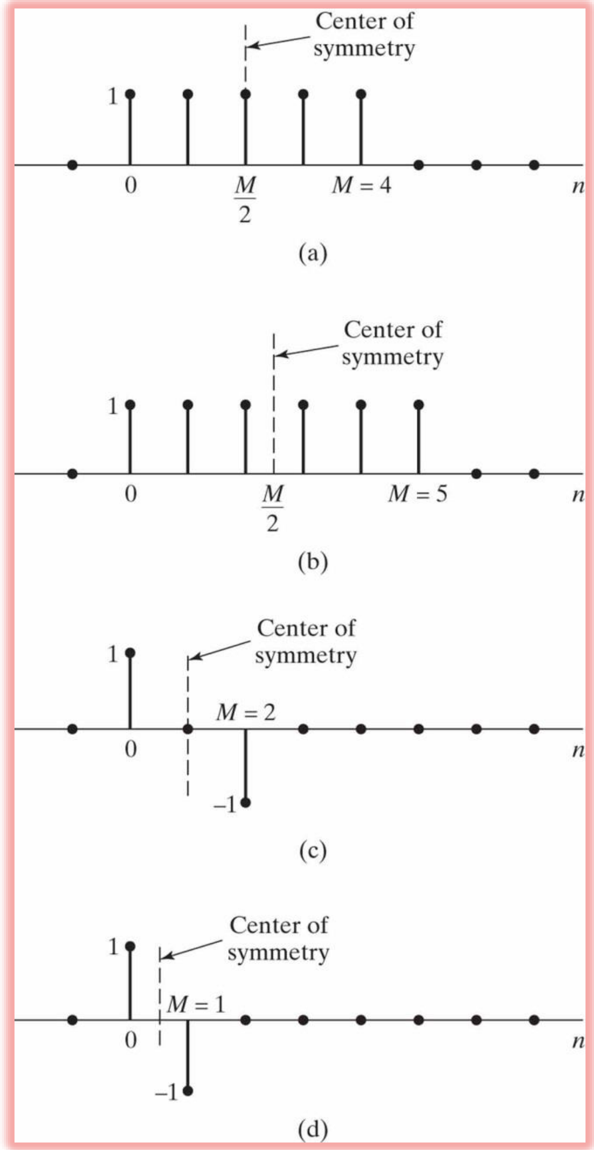
  $\xrightarrow{\text{generalize}} \quad H(e^{j\omega}) = A(e^{j\omega})e^{-j(\omega\alpha-\beta)}$

  - $A(e^{j\omega})$: a real function that may have negative values

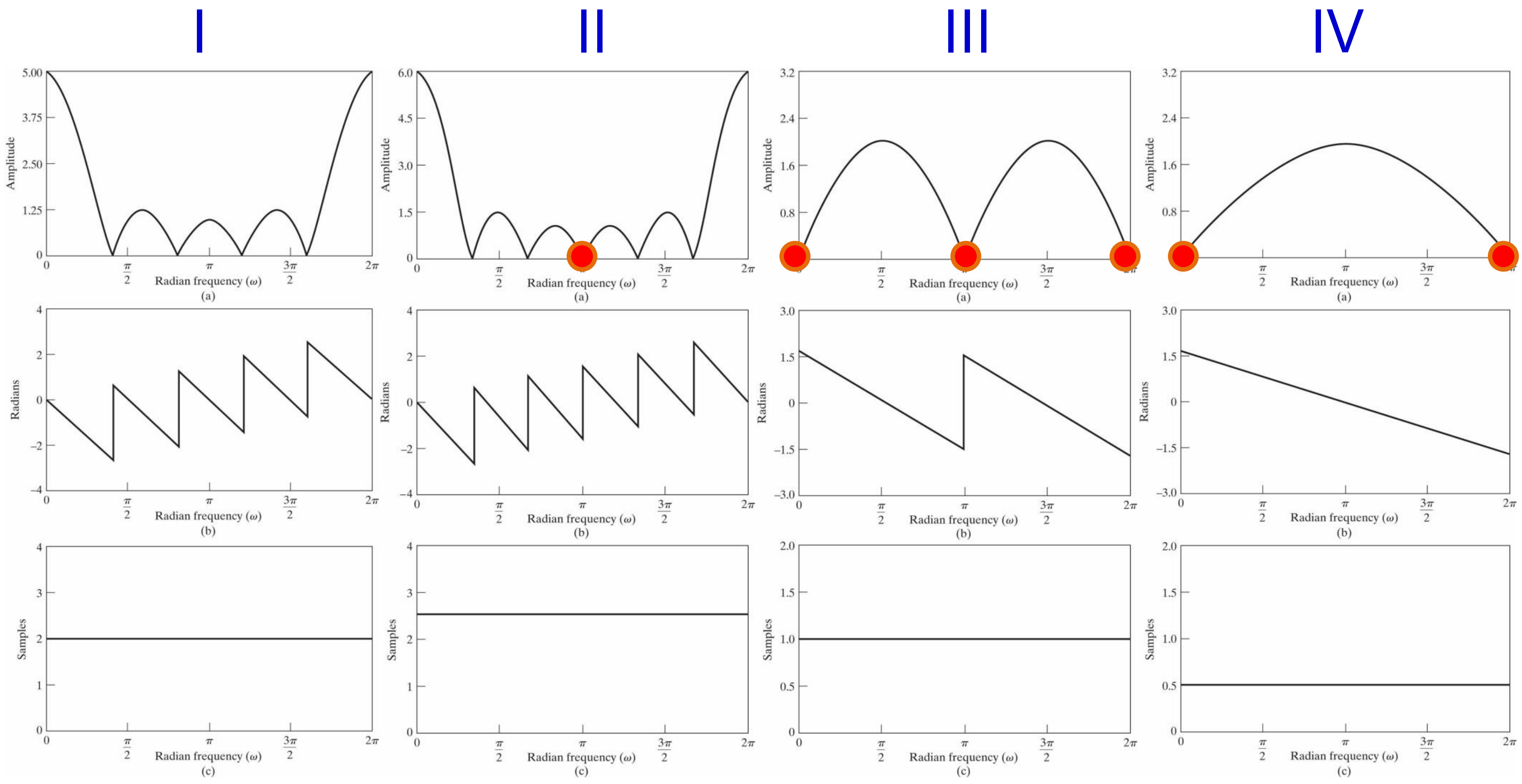
- Note it has a constant group delay

Four Typical Types of FIR Linear-Phase Systems

- Type I: $h[n] = h[M-n]$, M even
  - $H(e^{j\omega}) = e^{-\frac{j\omega M}{2}} \sum_{k=0}^{M/2} a[k] \cos \omega k$
- Type II: $h[n] = h[M-n]$, M odd
- Type III: $h[n] = -h[M-n]$, M even
  - $H(e^{j\omega}) = je^{-\frac{j\omega M}{2}} \sum_{k=0}^{M/2} c[k] \sin \omega k$
- Type IV: $h[n] = -h[M-n]$, M odd



Four Typical Types of FIR Linear-Phase Systems



Four Typical Types of FIR Linear-Phase Systems

- Zeroes appear in a quadruple manner $\{z_0, z_0^{-1}, z_0^*, (z_0^*)^{-1}\}$
- In types I and II
  - $H(z) = z^{-M}H(z^{-1})$
  - Type II has a zero at $z = -1$
- In types III and IV
  - $H(z) = -z^{-M}H(z^{-1})$
  - zero at $z = 1$
  - Type III has a zero at $z = -1$