Digital Signal Processing

Chap 8.
Discrete Fourier Transform

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Definition

Definition

- N-point input signal $x[n],\ 0 \le n \le N-1$

- Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

for each $0 \le k \le N-1$, where $W_N = e^{-j\frac{2\pi}{N}}$

- Inverse Discrete Fourier Transform (IDFT)

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

for each $0 \le n \le N-1$

DFT is a Lossless Representation

- $x[n] \overset{\text{DFT}}{\Longrightarrow} X[k] \overset{\text{IDFT}}{\Longrightarrow} y[n], \text{then } y[n] = x[n]$

Examples

- Ex 1) Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \leq n \leq N-1 \end{cases}$$

- Ex 2) Consider the length-N sequence

$$g[n] = \cos\left(\frac{2\pi r n}{N}\right)$$

where r is an integer between 1 and $N-1$

Matrix Representation of DFT

Forward Transform

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \qquad \text{or}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Matrix Representation of DFT

Inverse Transform

$$\mathbf{x}_N = \mathbf{W}_N^{-1}\mathbf{X}_N \qquad \text{or}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

- DFT can be interpreted as an invertible matrix
- The forward and inverse matrices are related by

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^{*}$$

Relationships between DFT and DTFT

DFT and DTFT

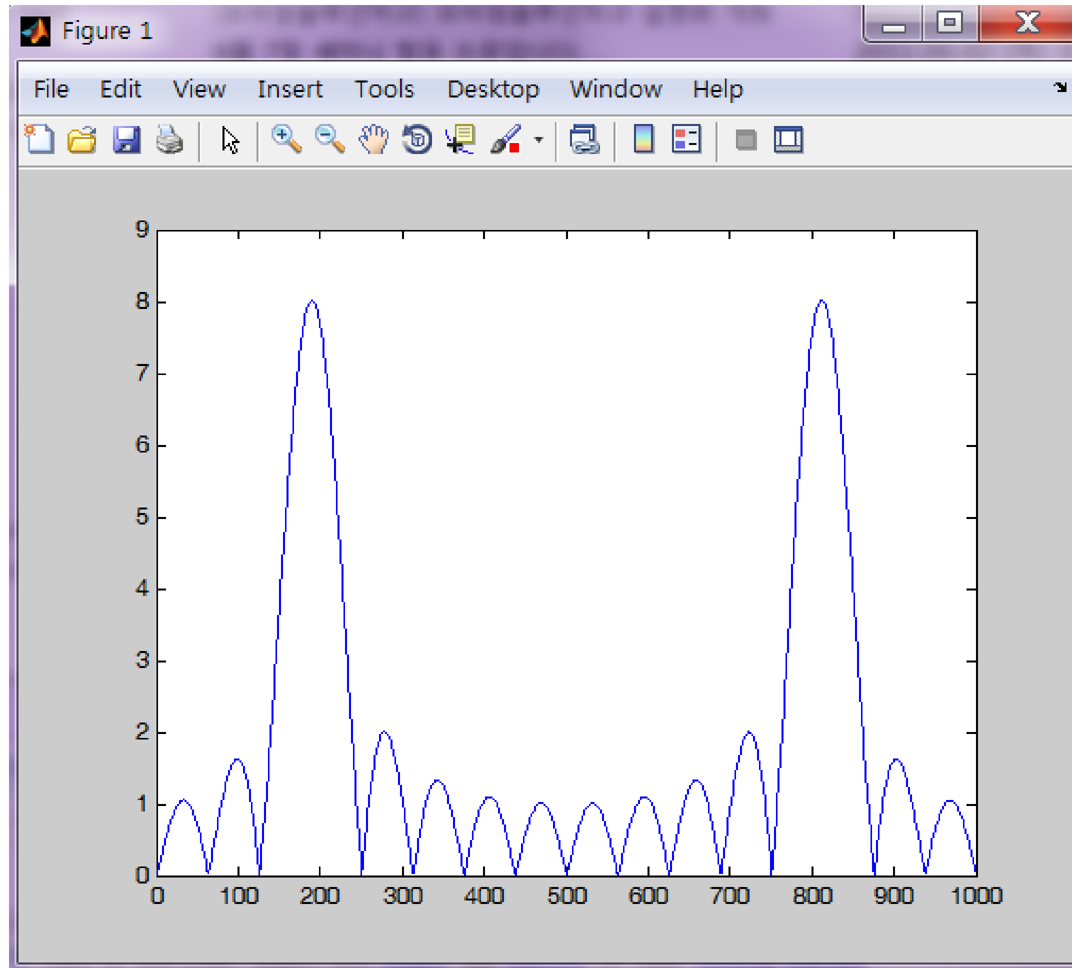
- Let $X(e^{j\omega})$ denote the DTFT of $x[n],\ 0 \le n \le N-1$, then

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

  - $X[k]$ is the set of frequency samples of the DFTF $X(e^{j\omega})$ of the length-N sequence at N equally spaced frequencies
  - Thus, $X[k]$ is also a frequency-domain representation of the sequence $x[n]$

DFT and DTFT

- DTFT of a finite-length sequence can be plotted with high precision using DFT
- Ex) DTFT of $x[n] = \cos\left(\frac{6\pi n}{16}\right), \quad 0 \le n \le 15$

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>> for i=1:16, x(i)=cos(6 * pi * i/16);, end
>> mag = abs(fft(x, 1000));
>> plot(mag)
```

Circular Convolution Theorem

Extensions are Periodic

- The extension of x is periodic with period N

$$x[n+N] = x[n]$$

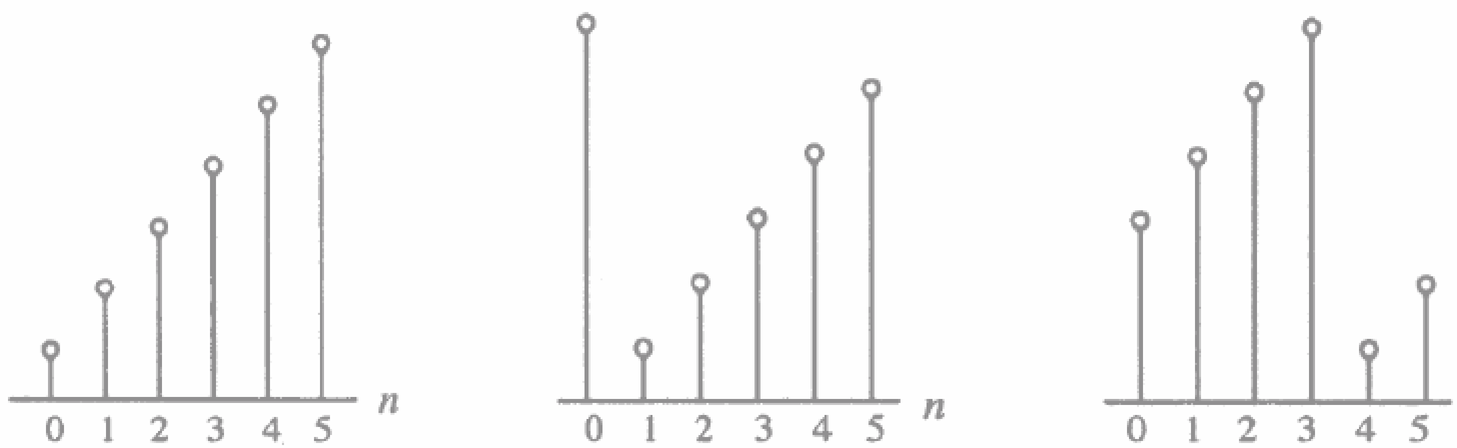
- Similarly, the extension of X is periodic with period N

$$X[k+N] = X[k]$$

Extensions are Periodic

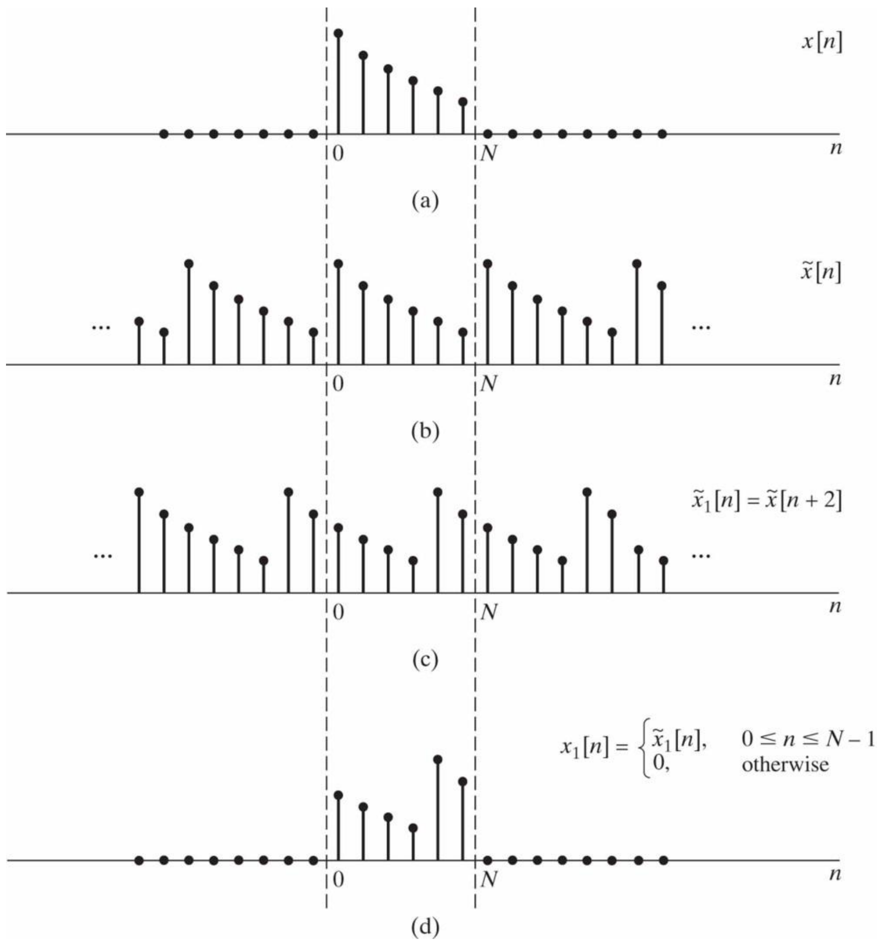
- $x[n]$ should be understood as $x[(n)_N]$
- $X[k]$ should be understood as $X[(k)_N]$
- Hence, when dealing with finite-length sequences, "shift" to the right by n_0 should be understood as the "**circular shift**."

$$x[n - n_0] = x[(n - n_0)_N]$$



$\boldsymbol{x[n]}$ $\boldsymbol{x[(n-1)_6]}$ $\boldsymbol{x[(n-4)_6]}$

Circular Shift



- A circular shift of an *N*-point sequence is equivalent to a linear shift of its periodic extension

Circular Shift

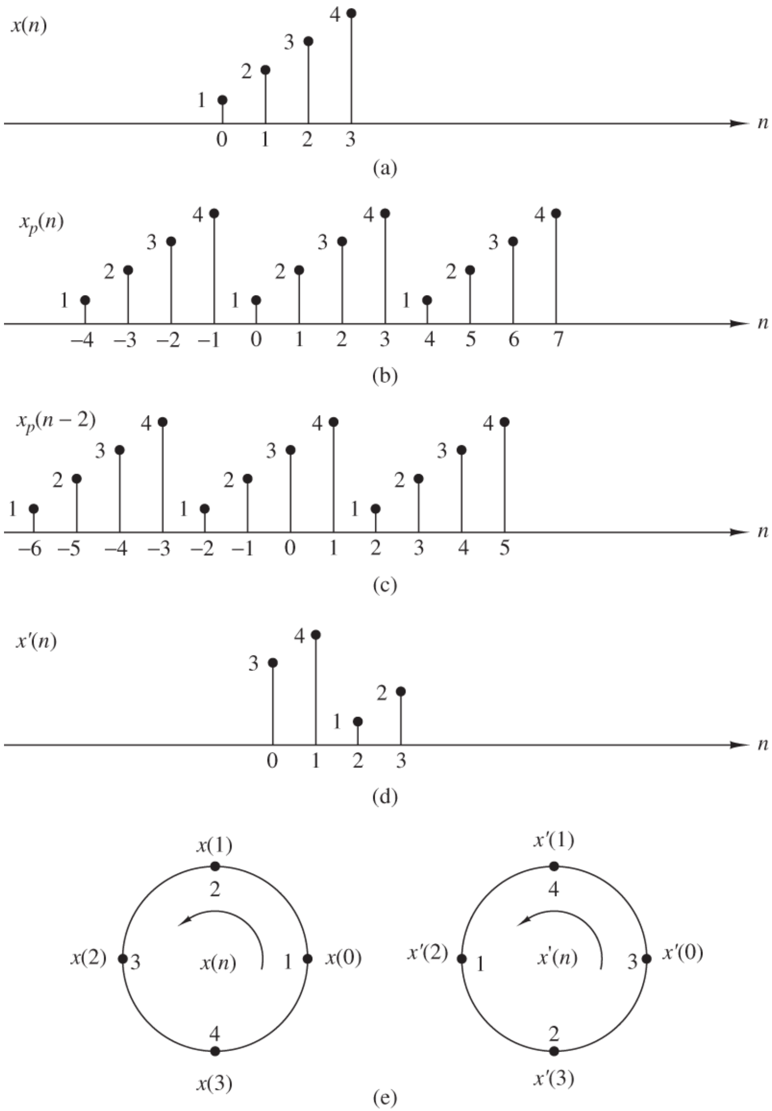


Figure 7.2.1 Circular shift of a sequence.

- A circular shift of an *N*-point sequence is equivalent to a linear shift of its periodic extension

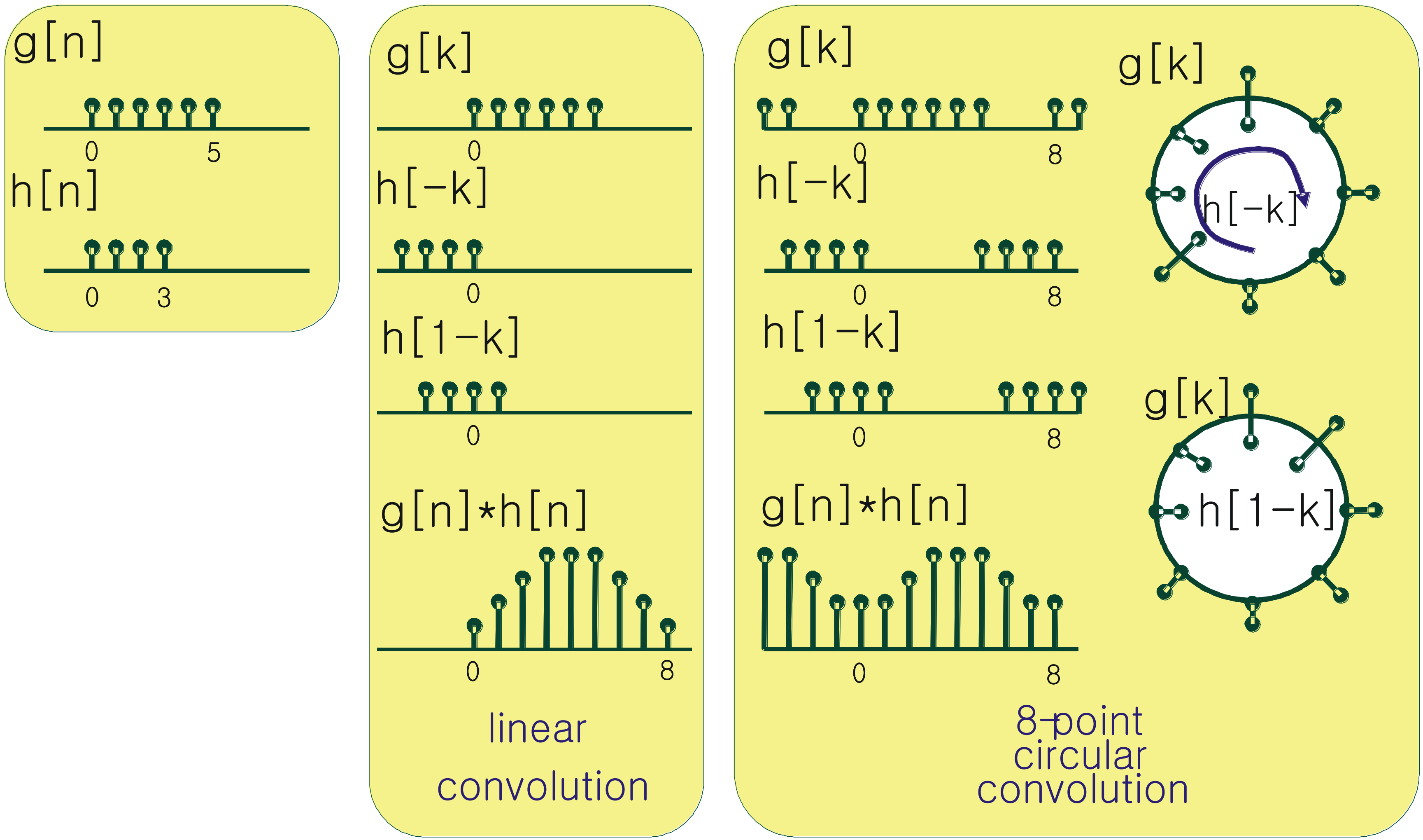
Linear Convolution vs. Circular Convolution

- Convolution of two N-point sequences $g[n]$ and $h[n]$

$$y[n] = g[n] * h[n] = \sum_{k=0}^{N-1} g[n-k]h[k] = \sum_{k=0}^{N-1} g[k]h[n-k]$$

  - Linear convolution
    - $g[n] = h[n] = 0 \quad$ for $n < 0$ or $n \geq N$
  - Circular convolution
    - $g[n + mN] = g[n]$
    - $h[n + mN] = h[n]$

Linear Convolution vs. Circular Convolution



N-Point Circular Convolution

$$y[n] = g[n] \circledast h[n] = \sum_{k=0}^{N-1} g[(n-k)_N]h[k] = \sum_{k=0}^{N-1} g[k]h[(n-k)_k]$$

- Ex) Circularly convolve $\{2, 1, 2, 1\}$ and $\{1, 2, 3, 4\}$

Using Circular Convolution to Obtain Linear Convolution

- Conditions
  - $g[n]$: **M-point** sequence, $g[n] = 0$ for $n < 0$ or $n > M - 1$
  - $h[n]$: **N-point** sequence, $h[n] = 0$ for $n < 0$ or $n > N - 1$
  - The **linear convolution of $\boldsymbol{g[n]}$ and $\boldsymbol{h[n]}$** generates **$(\boldsymbol{M + N - 1})$-point** sequence, $g[n] * h[n] = 0$ for $n < 0$ or $n > M + N - 2$

- Procedures
  1. Zero padding $g[n]$ and $h[n]$ to yield $(M + N - 1)$-point sequence $g_p[n]$ and $h_p[n]$.
  2. Obtain $(M + N - 1)$-point circular convolution of $g_p[n]$ and $h_p[n]$
  3. Result of Step 2 is equivalent to the linear convolution of $g[n]$ and $h[n]$

Circular Convolution Theorem

- $g[n] \circledast h[n] \overset{\text{DFT}}{\Longleftrightarrow} G[k]H[k]$

Additional Properties of DFT

Real-Valued Sequence $x[n]$

- $X^*[k] = X[-k] = X[N-k]$

Time Reversal

- $x[(-n)_N] = x[N-n] \overset{\text{DFT}}{\Longleftrightarrow} X[(-k)_N] = X[N-k]$

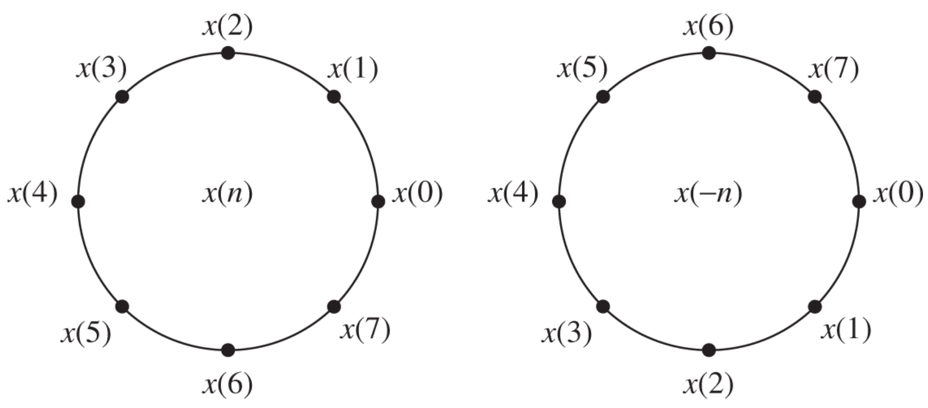


Figure 7.2.3 Time reversal of a sequence.

Circular Shift

- $x[(n-l)_N] \overset{\text{DFT}}{\Longleftrightarrow} X[k]W_N^{kl}$
- $x[n]W_N^{-nl} \overset{\text{DFT}}{\Longleftrightarrow} X[(k-l)_N]$

Multiplication of Two Sequences

- $x[n]y[n] \overset{\text{DFT}}{\Longleftrightarrow} \frac{1}{N} X[k] \circledast Y[k]$

Parseval's Theorem

- $\sum_{n=1}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=1}^{N-1} |X[k]|^2$