Chapter 15. Power Series and Taylor Series

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The contents herein are based on the book "Advanced Engineering Mathematics" by E. Kreyszig and only for the course KEEE202, Korea University.

I. Sequences and Series

We denote a sequence of complex numbers, $z_1, z_2, z_3, \ldots$, by $\{z_n\}$.

⋆ Convergence

A sequence is called convergent

$$\lim_{n\to\infty} z_n = c,$$

if for any $\epsilon > 0$, there exists N such that

$$\|z_n - c\| < \epsilon \qquad \text{for all } n > N.$$

A sequence is said to be divergent, if it is not convergent.

Let us consider a series

$$\sum_{m=1}^{\infty} z_m = z_1 + z_2 + \cdots$$

Its partial sum is defined by

$$s_n = \sum_{m=1}^{n} z_m.$$

The series is called convergent if $\{s_n\}$ converges.

⋆ Theorem: If a series $z_1 + z_2 + \cdots$ converges, then

$$\lim_{n\to\infty} z_n = 0.$$

This theorem implies that if

$$\lim_{n\to\infty} z_n \neq 0,$$

the series diverges. However, it does not imply that if

$$\lim_{n\to\infty} z_n = 0,$$

the series converges.

⋆ Cauchy's convergence test:

A series $z_1 + z_2 + \cdots$ is convergent, if and only if for any $\epsilon > 0$ we can find N such that

$$\|z_{n+1} + \cdots + z_{n+p}\| < \epsilon$$

for all $n > N$ and p.

A series $z_1 + z_2 + \cdots$ is called absolutely convergent, if

$$\sum_{m=1}^{\infty} \|z_m\|$$

is convergent.

⋆ Theorem: If a series is absolutely convergent, it is convergent.

Note that Cauchy's convergence test is not helpful in practice, though it is very useful in proving many theorems and properties. Let us see how to test whether a given sequence is convergent or not in practice.

⋆ Practical Method 1 (Comparison Test): If $\|z_i\| \leq b_i$ and $\sum b_i$ converges, then $\sum z_i$ converges.

⋆ Practical Method 2 (Ratio Test): $\sum z_n$ converges, if

$$\|\frac{z_{n+1}}{z_n}\| \leq q < 1 \qquad \text{for } n > N,$$

where N is any fixed number.

Variation of Ratio Test: Let $L = \lim_{n\to\infty} \|\frac{z_{n+1}}{z_n}\|$.

- If $L < 1$, the series converges.
- If $L > 1$, it diverges.
- If $L = 1$, the test fails.

⋆ Practical Method 3 (Root Test): $\sum z_n$ converges, if

$$\sqrt[n]{\|z_n\|} \leq q < 1 \qquad \text{for } n > N,$$

where N is any fixed number.

Variation of Root Test: Let $L = \lim_{n\to\infty} \sqrt[n]{\|z_n\|}$.

- If $L < 1$, the series converges.
- If $L > 1$, it diverges.
- If $L = 1$, the test fails.

II. Power Series

Let us consider a series, given by

$$\sum_{n=0}^{\infty} \underbrace{a_n}_{\textbf{Coefficient}} (\overbrace{z}^{\textbf{Complex variable}} - \underbrace{z_0}_{\textbf{Center}})^n = f(z)$$

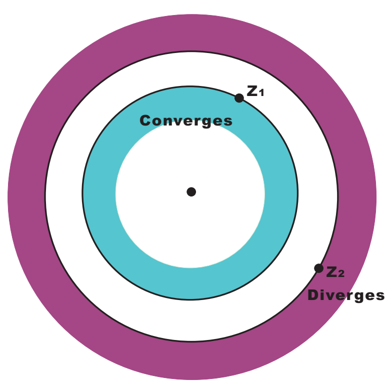
It contains a variable z, so it is a function $f(z)$. Its convergence also depends on z.

⋆ Ex 1 : $\sum_{n=0}^{\infty} z^n$ converges when $\|z\| < 1$

⋆ Ex 2 : $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges everywhere.

⋆ Ex 3 : $\sum_{n=0}^{\infty} n! z^n$ converges only when $z = 0$..

⋆ Convergence of a Power Series:



- It converges at the center z_0
- If it converges at a point z_1, it converges for every z closer to z_0 than z_1, *i.e.*, $\|z - z_0\| < \|z_1 - z_0\|$.
- If it diverges at z_2, it diverges for every z farther away from z_2, *i.e.*, $\|z - z_0\| > \|z_2 - z_0\|$.

$\Rightarrow$ These properties imply that the region of convergence has always a circular shape.

Proof) Skipped.

⋆ Radius of Convergence R:

The radius of convergence R is defined such that

- the series converges within $\|z - z_0\| < R$,
- it diverges when $\|z - z_0\| > R$,
- it may converge or diverge on the circle.

It is a convention to set $R = \infty$ if the series converges everywhere. Also, $R = 0$, if the series converges only when $z = z_0$.

⋆ Theorem (How to find R?):

Suppose that

$$\lim_{n\to\infty} \frac{\|a_{n+1}\|}{\|a_n\|} = L^*.$$

Then

$$R = \frac{1}{L^*}.$$

$$\begin{pmatrix} R = \infty & \text{if} & L^* = 0 \\ R = 0 & \text{if} & L^* = \infty \end{pmatrix}$$

⋆ Ex :

$$\sum_{n=1}^{\infty} \frac{(z+i)^n}{n^2}$$

⋆ Ex :

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}(z+2i)^n$$

III. Property of Power Series

We consider the properties of a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

These properties can be straightforwardly extended to the general case of

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n.$$

1. (Continuity at the center) If the power series has the radius of convergence $R > 0$, it is continuous at $z = 0$.

$$\lim_{z \to 0} f(z) = f(0) = a_0.$$

2. (Unique representation) If $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = b_0 + b_1 z + b_2 z^2 + \cdots$, then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, $\cdots$.

3. (Addition) If $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with R_1 and $g(z) = b_0 + b_1 z + b_2 z^2 + \cdots$ with R_2, then $f(z) + g(z) = (a_0 + b_0) + (a_1 + b_1) z + (a_2 + b_2) z^2 + \cdots \qquad$ with $R \geq \min(R_1, R_2)$.

4. (Multiplication)

$$\begin{aligned} f(z) \cdot g(z) &= a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots \\ &= \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) z^n \qquad \text{with } R \geq \min(R_1, R_2). \end{aligned}$$

5. (Differentiation) $f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad$ with $R = R_1$.

6. (Integration) $\int f(z) dz = a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \qquad$ with $R = R_1$.

7. A power series with a nonzero radius of convergence R represents an analytic function on the domain $\{z : \|z\| < R\}$.

⋆ Ex 1 : Find the radius of convergence of $\sum_{n=2}^{\infty} \frac{n(n-1)}{3^n} (z - 2i)^n$

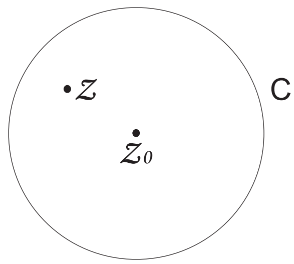
⋆ Ex 2: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is odd, show that $a_n = 0$ for all even n.

IV. Taylor Series

⋆ Theorem: Every analytic function can be represented by a power series

$$f(z) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Sketch of proof)



Note that for z^* on C, we have

$$\begin{aligned}
\frac{1}{z^* - z} &= \frac{1}{z^* - z_0 - (z - z_0)} \\
&= \frac{1}{z^* - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{z^* - z_0}} \\
&= \frac{1}{z^* - z_0}\Big(1 + \frac{z - z_0}{z^* - z_0} + \Big(\frac{z - z_0}{z^* - z_0}\Big)^2 + \cdots\Big).
\end{aligned}$$

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z} dz^* \\
&= \frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^* - z_0} dz^* + \frac{(z - z_0)}{2\pi i}\oint_C \frac{1}{(z^* - z_0)^2} dz^* + \frac{(z - z_0)^2}{2\pi i}\oint_C \frac{1}{(z^* - z_0)^3} dz^* + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n. \qquad \Big(\because \frac{n!}{2\pi i}\oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^* = f^{(n)}(z_0)\Big)
\end{aligned}$$

⋆ A few Taylor series.

- $f(z) = \frac{1}{1-z} = 1 + z + z^2 + \cdots \qquad |z| < 1$
- $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$
- $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$
- $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$
- $\mathrm{Ln}(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots$

⋆ Examples:

- $\frac{1}{1+z^2}$
- $\arctan z$
- Represent $\frac{1}{c-z}$ in terms of $(z-z_0)^n$, where $c \neq z_0$.
- $\mathrm{Si}(z) = \int_0^z \frac{\sin z}{z} dz$