Gauss - Jordan Method to find A^{-1}

* If a matrix A is nonsingular (invertible), we can make it diagonal using elementary matrices and permutation matrices. For example

$$E_4 P_2 E_3 P_1 E_2 E_1 A = D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Then,

$$D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 A = I.$$

Therefore

$$A^{-1} = D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 .$$

If we start the forward elimination with

$$[\,A \quad I\,],$$

the result is

$$D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 [A \quad I]$$

$$= [\,I \quad A^{-1}\,].$$

To summarize, carry out the elimination with $[A \quad I]$ to make the identity in the left side, i.e, $[I \quad B]$.

Then B is the inverse of A.

Example) $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} ① \\ ② \\ ③ \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ ② - 2 \times ① \\ ③ + ① \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} ① + \frac{1}{8} \times ② \\ \\ ② + ③ \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & -\frac{5}{8} & -\frac{3}{4} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} ① - \frac{3}{4} ③ \\ ② + 2 \times ③ \\ \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} ① \times \frac{1}{2} \\ ② \times -\frac{1}{8} \\ \end{array}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

Inverses

A matrix A is invertible if there exists B such that $BA = I$ and $AB = I$. There is at most one such B, called the inverse of A and denoted by A^{-1}!

* $BA = I, \quad AC = I \quad \Rightarrow \quad B = C$

  $\left(\because \quad B = B(AC) = (BA)C = C \right)$

* $(A^{-1})^{-1} = A$

* $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{bmatrix}$

* $(AB)^{-1} = B^{-1}A^{-1}$

  $\left(\because \quad (AB)B^{-1}A^{-1} = ABB^{-1}A^{-1} = AA^{-1} = I \right.$

  $\left. \quad B^{-1}A^{-1}(AB) = B^{-1}A^{-1}AB = BB^{-1} = I \right)$

Transposes

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}$$

Def) $(A^T)_{ij} = A_{ji}$

* $(AB)^T = B^T A^T$

* $(A^{-1})^T = (A^T)^{-1}$

  $\left[\because \quad (AA^{-1})^T = (A^{-1})^T A^T = I \right.$

  $\left. \quad \Rightarrow (A^T)^{-1} = (A^{-1})^T \right]$