CONTEXT DEPENDENT CLASSIFICATION

- We have a sequence of observations (feature vectors) that are depedent

- Remember: Bayes rule

$$P(\omega_i|\underline{x}) > P(\omega_j|\underline{x}),\ \forall j \neq i$$

- Here: The class to which a feature vector belongs depends on:
  - Its own value
  - The values of the other features
  - An existing relation among the various classes

- This interrelation demands the classification to be performed simultaneously for all available feature vectors

- Thus, we will assume that the training vectors $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ occur in sequence, one after the other and we will refer to them as **observations**

❖ The Context Dependent Bayesian Classifier

- Let $X : \{\underline{x}_1, \underline{x}_2, ..., \underline{x}_N\}$
- Let $\omega_i, \ i = 1,2,..., M$
- Let Ω_i be a sequence of classes, that is

$$\Omega_i : \omega_{i1}\ \omega_{i2}\ ...\ \omega_{iN}$$

There are M^N of those

- Thus, the Bayesian rule can equivalently be stated as

$$X \to \Omega_i : \ P(\Omega_i | X) > P(\Omega_j | X) \ \ \forall i \neq j, \ \ i, j = 1,2,..., M^N$$

❖ Markov Chain Models (for class dependence)

$$P(\omega_{i_k} | \omega_{i_{k-1}}, \omega_{i_{k-2}}, ..., \omega_{i_1}) = P(\omega_{i_k} | \omega_{i_{k-1}})$$

❖ NOW remember:

$$P(\Omega_i) = P(\omega_{i_1}, \omega_{i_2}, ..., \omega_{i_N}) =$$

$$= P(\omega_{i_N} \left| \omega_{i_{N-1}}, ..., \omega_{i_1} \right.).$$

$$P(\omega_{i_{N-1}} \left| \omega_{i_{N-2}}, ..., \omega_{i_1} \right.) ... P(\omega_{i_1})$$

or

$$P(\Omega_i) = (\prod_{k=2}^{N} P(\omega_{i_k} \left| \omega_{i_{k-1}} \right.)) P(\omega_{i_1})$$

❖ Assume:

➢ $\underline{x}_i$ depends only on its class

$$p(X \left| \Omega_i \right.) = \prod_{k=1}^{N} p(\underline{x}_k \left| \omega_{i_k} \right.)$$

- From the above, the Bayes rule is readily seen to be equivalent to:

$$P(\Omega_i|X)(><)P(\Omega_j|X)$$

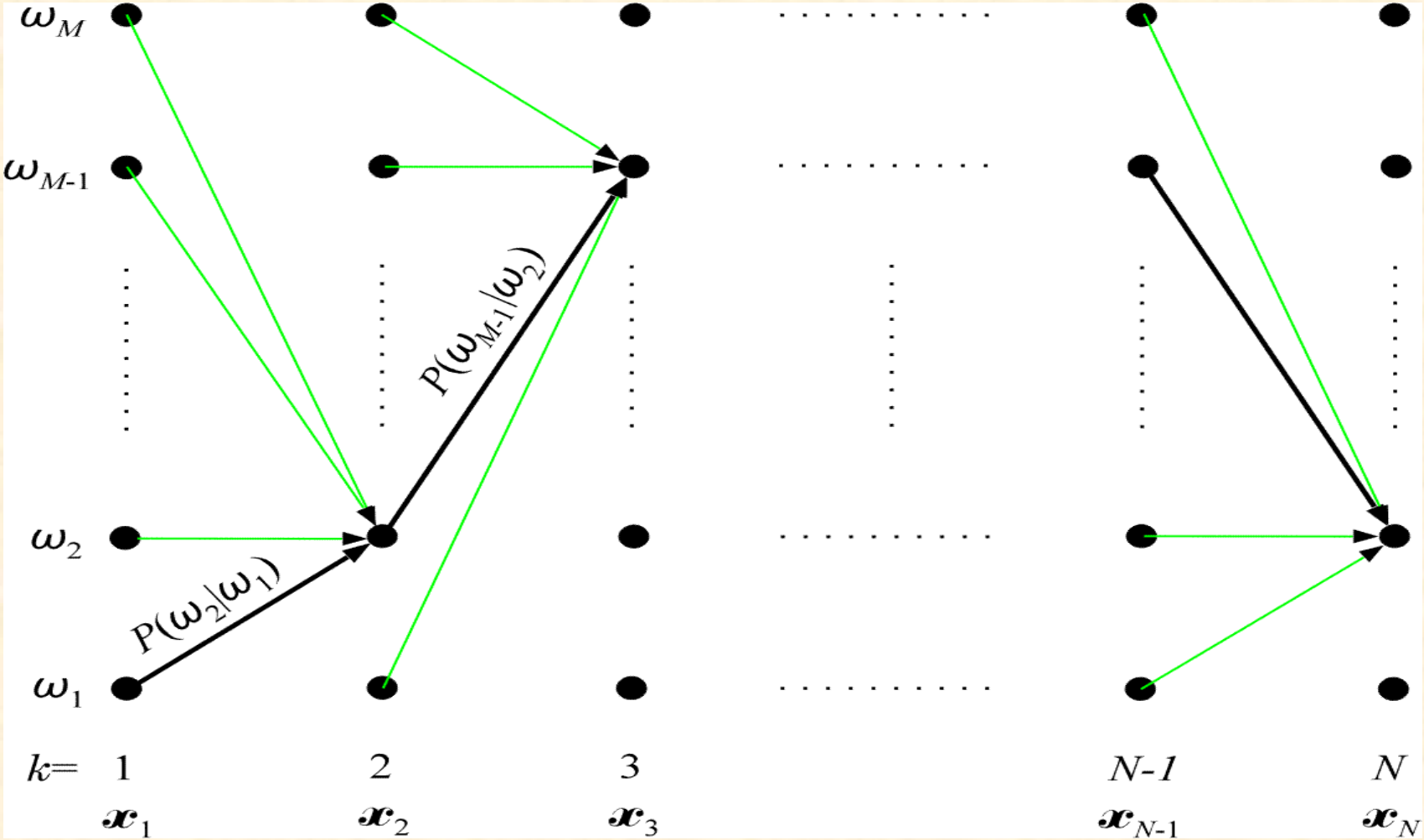
$$P(\Omega_i)\,p(X|\Omega_i)(><)P(\Omega_j)\,p(X|\Omega_j)$$

that is, it rests on

$$p(X|\Omega_i)P(\Omega_i) = P(\omega_{i_1})\,p(\underline{x}_1|\omega_{i_1}).$$
$$\prod_{k=2}^{N} P(\omega_{i_k}|\omega_{i_{k-1}})\,p(\underline{x}_k|\omega_{i_k})$$

- To find the above maximum in brute-force task we need $O(NM^N)$ operations!!

❖ The Viterbi Algorithm



- Thus, each Ω_i corresponds to one path through the trellis diagram. One of them is the optimum (e.g., black). The classes along the optimal path determine the classes to which ω_i are assigned.

- To each transition corresponds a cost. For our case

  - $\hat{d}(\omega_{i_k}, \omega_{i_{k-1}}) = P(\omega_{i_k} | \omega_{i_{k-1}}).$

    $$p(\underline{x}_k | \omega_{i_k})$$

  - $\hat{d}(\omega_{i_1}, \omega_{i_0}) \equiv P(\omega_{i_1}) p(\underline{x}_i | \omega_{i_1})$

  - $$\hat{D} = \prod_{k=1}^{N} \hat{d}(\omega_{i_k}, \omega_{i_{k-1}}) = p(X | \Omega_i) P(\Omega_i)$$

- Equivalently

$$\ln \hat{D} = \sum_{k=1}^{N} \ln \hat{d}(.,.) \equiv D = \sum_{k=1}^{N} d(.,.)$$

where,

$$d(\omega_{i_k}, \omega_{i_{k-1}}) = \ln \hat{d}(\omega_{i_k}, \omega_{i_{k-1}})$$

- Define the cost up to a node ,*k*,

$$D(\omega_{i_k}) = \sum_{r=1}^{k} d(\omega_{i_r}, \omega_{i_{r-1}})$$

- Bellman's principle now states

$$D_{\max}(\omega_{i_k}) = \max_{i_{k-1}} \left[D_{\max}(\omega_{i_{k-1}}) + d(\omega_{i_k}, \omega_{i_{k-1}}) \right]$$

$$i_k, i_{k-1} = 1,2,\ldots, M$$

$$D_{\max}(\omega_{i_0}) = 0$$

- The optimal path terminates at ω_{iN}^* :

$$\omega_{i_N}^* = \arg\max_{\omega_{i_N}} D_{\max}(\omega_{i_N})$$

  - Complexity $O(NM^2)$

❖ Channel Equalization

➢ The problem

- $x_k = f(I_k, I_{k-1}, \ldots, I_{k-n+1}) + n_k$
- $\underline{x}_k \equiv [x_k, x_{k-1}, \ldots, x_{k-l+1}]^T$
- $\underline{x}_k \rightarrow \hat{I}_k$ or $\hat{I}_{k-r}$

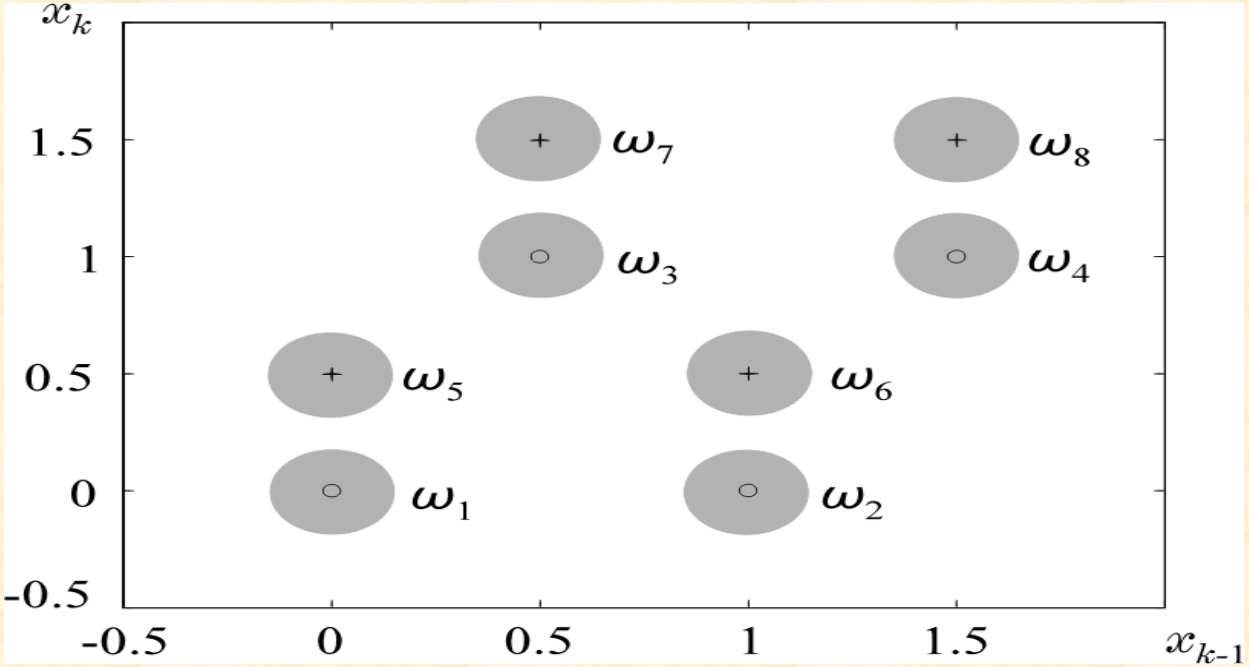
$$\underline{x}_k \rightarrow \boxed{\text{equalizer}} \rightarrow \hat{I}_{k-r}$$

➢ Example

- $x_k = 0.5I_k + I_{k-1} + n_k$
- $\underline{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}, l = 2$
- In $\underline{x}_k$ three input symbols are involved:

  I_k, I_{k-1}, I_{k-2}

I_k	I_{k-1}	I_{k-2}	x_k	x_{k-1}	
0	0	0	0	0	ω_1
0	0	1	0	1	ω_2
0	1	0	1	0.5	ω_3
0	1	1	1	1.5	ω_4
1	0	0	0.5	0	ω_5
1	0	1	0.5	1	ω_6
1	1	0	1.5	0.5	ω_7
1	1	1	1.5	1.5	ω_8



➢ Not all transitions are allowed

- $(I_k, I_{k-1}, I_{k-2}) = (0, 0, 1)$
- Then

$$(I_{k+1}, I_k, I_{k-1}) \begin{cases} (1, 0, 0) \\ (0, 0, 0) \end{cases}$$

- $\omega_2 \begin{cases} \omega_5 \\ \omega_1 \end{cases}$

$$P(\omega_2 | \omega_i) = \begin{cases} 0.5, i = 5, 1 \\ 0, \text{otherwise} \end{cases}$$

- In this context, ω_i are related to states. Given the current state and the transmitted bit, I_k, we determine the next state. The probabilities $P(\omega_i|\omega_j)$ define the state dependence model.

➢ The transition cost

- $$d(\omega_{i_k}, \omega_{i_{k-1}}) = d_{\omega_{i_k}}(\underline{x})$$

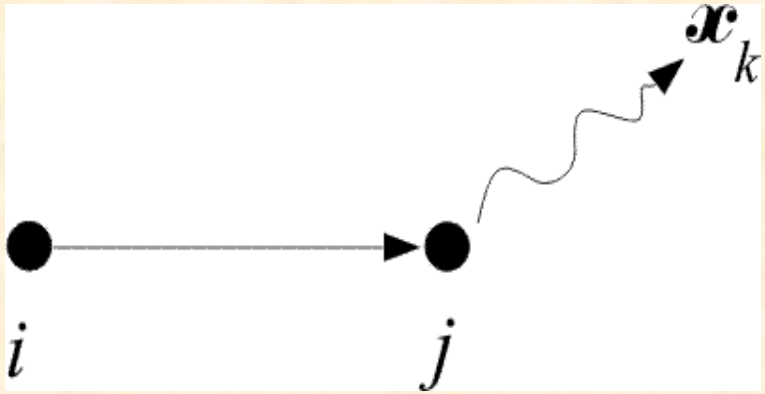
$$= \left\| \underline{x}_k - \underline{\mu}_{ik} \right\| = \left((\underline{x}_k - \underline{\mu}_{i_k})^T \sum\nolimits_{i_k}^{-1} (\underline{x}_k - \underline{\mu}_{i_k}) \right)^{\frac{1}{2}}$$

for all allowable transitions

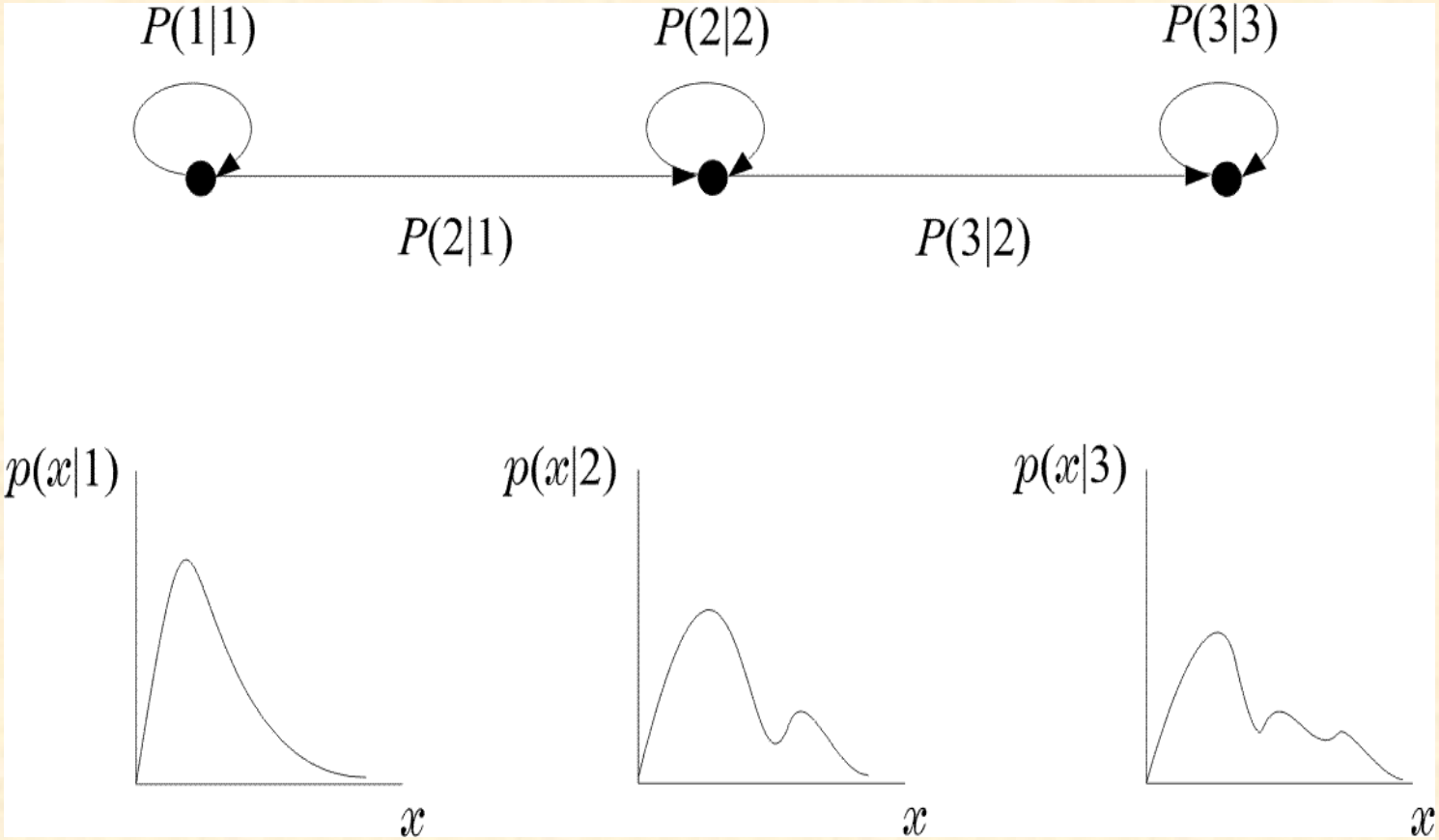
❖ Hidden Markov Models

- In the channel equalization problem, the states are observable and can be "learned" during the training period
- Now we shall assume that states are not observable and can only be inferred from the training data
- Applications:
  - Speech and Music Recognition
  - OCR
  - Blind Equalization
  - Bioinformatics

- An HMM is a stochastic finite state automaton, that generates the observation sequence, $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$
- We assume that: The observation sequence is produced as a result of **successive** transitions between states, upon arrival at a state:



- This type of modeling is used for nonstationary stochastic processes that undergo distinct transitions among a set of different stationary processes.



- The single coin case: Assume that a coin is tossed behind a curtain. We only know the outcome, i.e., H or T. Assume the two states to be:

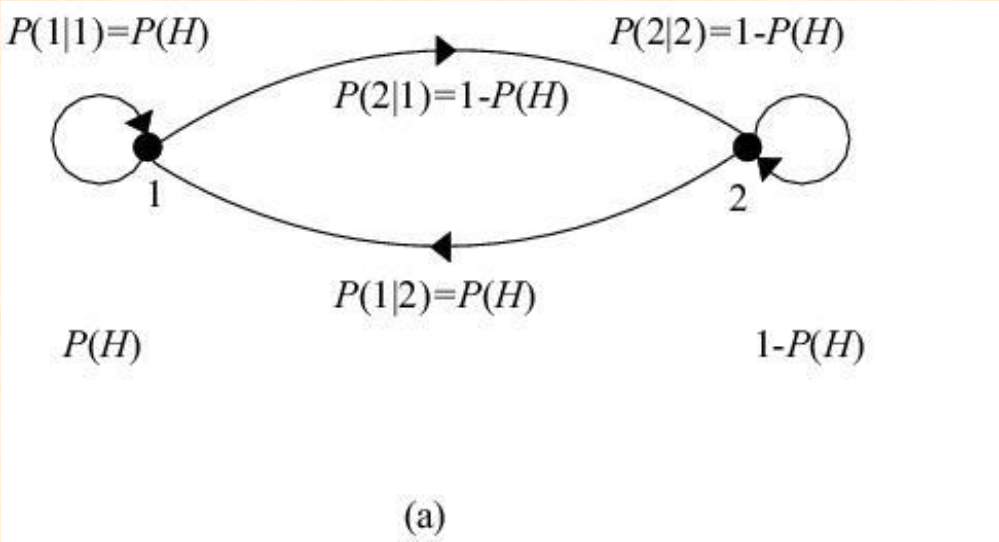
$$S = 1 \rightarrow H$$

$$S = 2 \rightarrow T$$

This is also an example of a random experiment with observable states. The model is characterized by a single parameter, e.g., $P(H)$. Note that

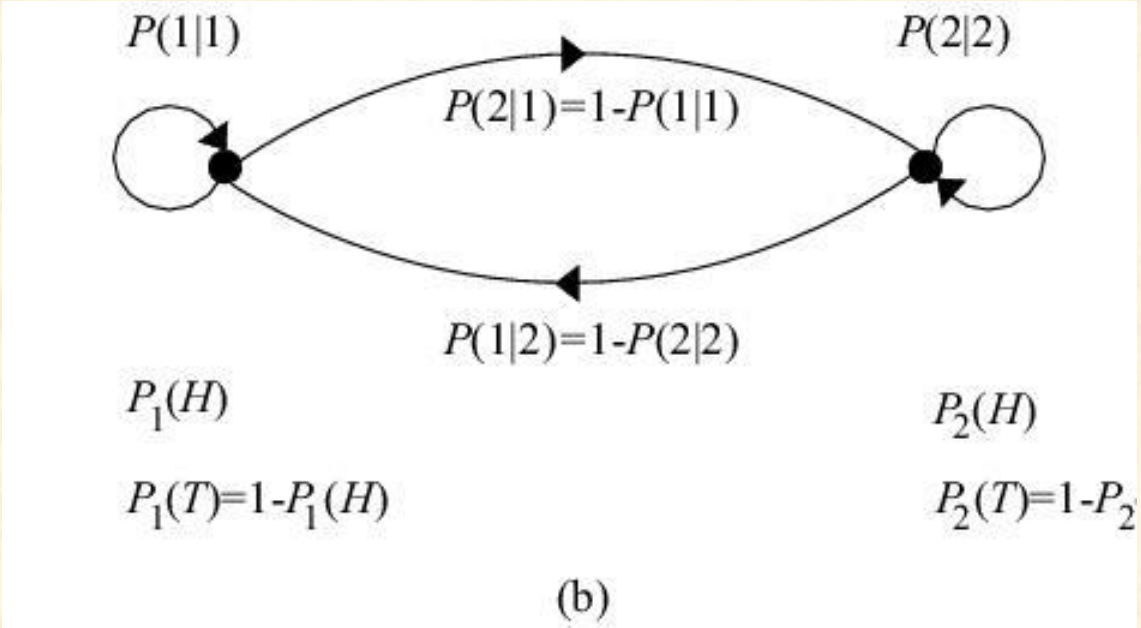
$$P(1|1) = P(H)$$

$$P(2|1) = P(\mathrm{T}) = 1 - P(H)$$

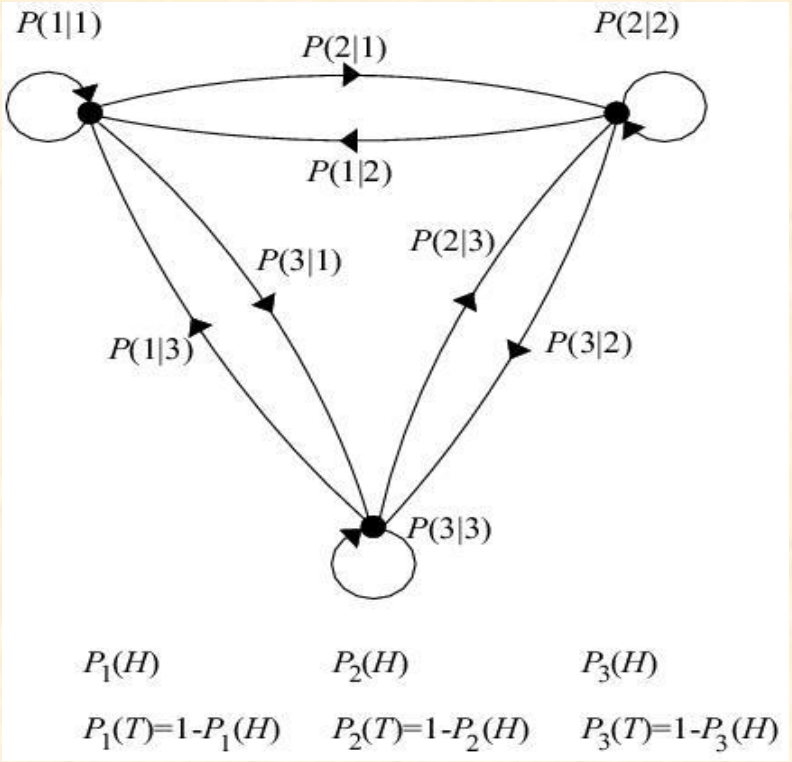


- Examples of HMM:
  - Two-coins case: We observe a sequence of H or T. However, we have no access to know which coin was tossed. Identify one state for each coin. States are not observable. H or T can be emitted from either state. The model depends on four parameters.

$$P_1(H),\ P_2(H),$$
$$P(1|1),\ P(2|2)$$



- Three-coins case:



- Note that in all previous examples, specifying the model is equivalent to knowing:
  - The probability of each observation (H,T) to be emitted from each state.
  - The transition probabilities among states: $P(i|j)$.

- A general HMM model is characterized by the following set of parameters
  - K, number of states
  - $P(i|j), i, j = 1,2,..., K$
  - $p(\underline{x}|i), i = 1,2,..., K$
  - $P(i), i = 1,2,..., K$, initial state probabilities, $P(.)$

That is:

$$S = \{P(i|j),\ p(\underline{x}|i), P(i), K\}$$

- What is the problem in Pattern Recognition
  - Given M reference patterns, each described by an HMM, find the parameters, S, for each of them (training)
  - Given an unknown pattern, find which one among the M known patterns matches it best (recognition)

➢ Recognition: Any path method

- Assume M models (M classes or patterns).
- A sequence of observations, X, is given.
- Decide in favor of the model S^* (from the M available) according to the Bayes rule

$$S^* = \arg\max_S P(S|X)$$

for equiprobable patterns

$$S^* = \arg\max_S p(X|S)$$

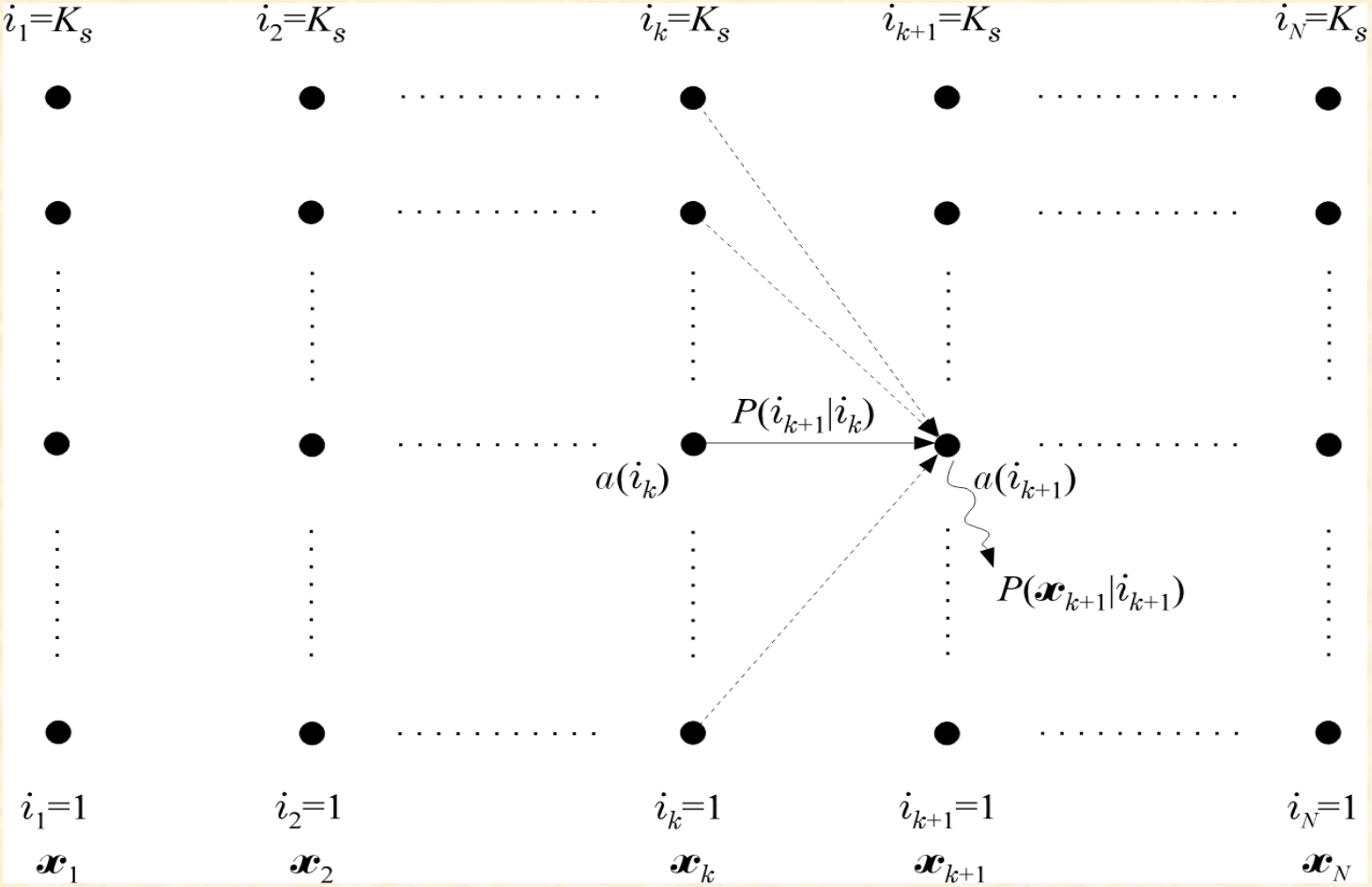
- For each model S there is more than one possible sets of successive state transitions Ω_i, each with probability $P(\Omega_i|S)$

Thus:
$$P(X|S) = \sum_i p(X, \Omega_i|S)$$
$$= \sum_i p(X|\Omega_i, S) P(\Omega_i|S)$$

- For the efficient computation of the above DEFINE
  - $\alpha(i_{k+1}) = p(\underline{x}_1, \ldots, \underline{x}_{k+1}, i_{k+1}|S)$

$$= \sum_{i_k} \alpha(i_k) \; P(i_{k+1}|i_k)\, p(\underline{x}_{k+1}|i_{k+1})$$

History — Local activity



- Observe that

$$P(X|S) = \sum_{i_N=1}^{K_S} \alpha(i_N)$$

Compute this for each S

- Some more quantities
  - $$\beta(i_k) = p(\underline{x}_{k+1}, \underline{x}_{k+2}, \ldots, \underline{x}_N \,|\, i_k, S)$$
    $$= \sum_{i_{k+1}} \beta(i_{k+1}) P(i_{k+1} \,|\, i_k)\, p(\underline{x}_{k+1} \,|\, i_{k+1})$$
  - $$\gamma(i_k) = p(\underline{x}_1, \ldots, \underline{x}_N, i_k \,|\, S)$$
    $$= \alpha(i_k)\beta(i_k)$$

- Training
  - The philosophy:

    Given a training set X, known to belong to the specific model, estimate the unknown parameters of S, so that the **output** of the model, e.g.

$$p(X|S) = \sum_{i_{N=1}}^{K_S} \alpha(i_N)$$

    to be maximized

- This is an ML parameter estimation problem

- Assumption: Data $\underline{x}$ discrete

$$\underline{x} \in \{1,2,\ldots,r\} \Rightarrow p(\underline{x}|i) \equiv P(\underline{x}|i)$$

- Definitions:

  - $$\xi_k(i,j) = \frac{\alpha(i_k = i)P(j|i)P(\underline{x}_{k+1}|j)\beta(i_{k+1} = j)}{P(X|S)}$$

  - $$\gamma_k(i) = \frac{\alpha(i_k = i)\beta(i_k = i)}{P(X|S)}$$

➢ The Algorithm:

- Initial conditions for all the unknown parameters.
  Compute $P(X|S)$

- Step 1: From the current estimates of the model parameters reestimate the new model S from

  – $\overline{P}(j|i) = \dfrac{\sum_{k=1}^{N-1} \xi_k(i,j)}{\sum_{k=1}^{N-1} \gamma_k(i)}$ $\left(= \dfrac{\#\text{ of transitions from } i \text{ to } j}{\#\text{ of transitions from } i} \right)$

  – $\overline{P}_{\underline{x}}(r|i) = \dfrac{\sum_{k=1 \text{ and } \underline{x} \to r)}^{N} \gamma_k(i)}{\sum_{k=1}^{N} \gamma_k(i)}$

  – $\overline{P}(i) = \gamma_1(i)$

- Step 3: Compute $P(X|\overline{S})$. If $P(X|\overline{S}) - P(X|S) > \varepsilon$, $S = \overline{S}$ go to step 2. Otherwise stop

- Remarks:
  - Each iteration improves the model

    $$\overline{S}: P(X|\overline{S}) > P(X|S)$$

  - The algorithm converges to a local maximum
  - The algorithm is an implementation of the EM algorithm