Chapter 1
Experiments, Models, and Probabilities

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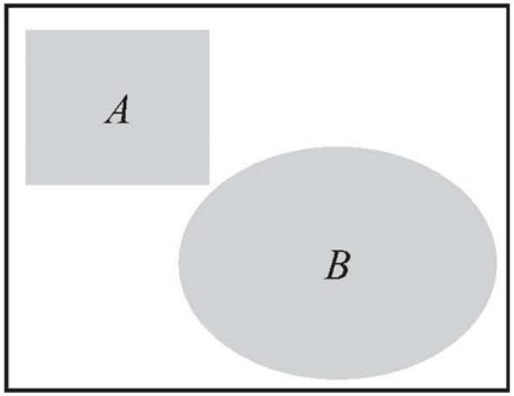
Section 1.1

Set Theory

- Probability theory is based on set theory
- It is enough to understand the terminology for basic set operations

1.1 Comment: Mutually Exclusive Sets

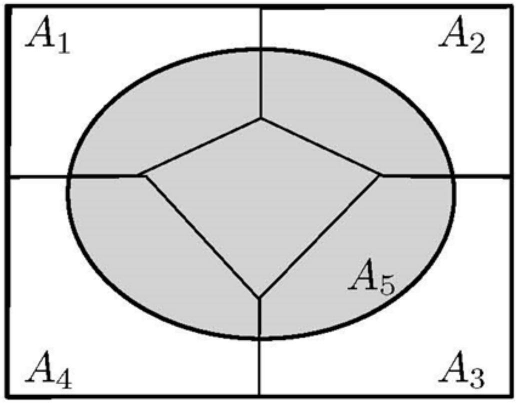


A collection of sets $A_1, \ldots, A_n$ is *mutually exclusive* if and only if

$$A_i \cap A_j = \varnothing, \qquad i \neq j. \tag{1.1}$$

The word *disjoint* is sometimes used as a synonym for mutually exclusive.

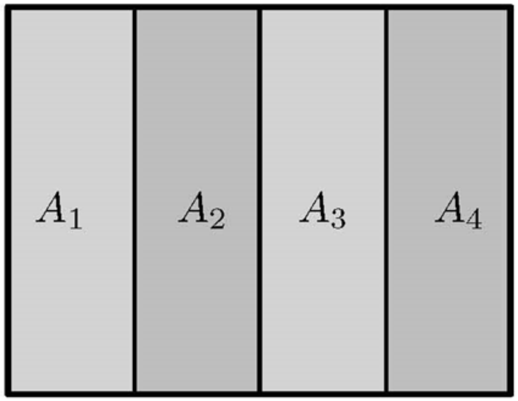
1.1 Comment: Collectively Exhaustive Sets



A collection of sets $A_1, \ldots, A_n$ issets,collectively exhaustive *collectively exhaustive* if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_n = S. \tag{1.2}$$

1.1 Comment: Partitions



A collection of sets $A_1, \ldots, A_n$ is a *partition* if it is both mutually exclusive and collectively exhaustive.

Example 1.1 Problem

Phonesmart offers customers two kinds of smart phones, Apricot (A) and Banana (B). It is possible to buy a Banana phone with an optional external battery E. Apricot customers can buy a phone with an external battery (E) or an extra memory card (C) or both. Draw a Venn diagram that shows the relationship among the items A,B,C and E available to Phonesmart customers.

Section 1.2

Applying Set Theory to Probability

- In this section, we define
  - experiment
  - outcome
  - event
  - sample space

1.1 Comment: Experiments

An experiment consists of a *procedure* and *observations*. There is uncertainty in what will be observed; otherwise, performing the experiment would be unnecessary. Some examples of experiments include

1. Flip a coin. Did it land with heads or tails facing up?

2. Walk to a bus stop. How long do you wait for the arrival of a bus?

3. Give a lecture. How many students are seated in the fourth row?

Example 1.2

An experiment consists of the following procedure, observation, and model:

- Procedure: Monitor activity at a Phonesmart store.
- Observation: Observe which type of phone (Apricot or Banana) the next customer purchases.
- Model: Apricots and Bananas are equally likely. The result of each purchase is unrelated to the results of previous purchases.

- A model makes it easier to study and analyze the experiment
- It is an approximation of the exact experiment

Example 1.3

Monitor the Phonesmart store until three customers purchase phones. Observe the sequence of Apricots and Bananas.

Example 1.4

Monitor the Phonesmart store until three customers purchase phones. Observe the number of Apricots.

Definition 1.1 Outcome

An outcome *of an experiment is any possible observation of that experiment.*

Definition 1.2 Sample Space

The sample space *of an experiment is the finest-grain, mutually exclusive, collectively exhaustive set of all possible outcomes.*

Example 1.5

- The sample space in Example 1.2 is $S = \{a, b\}$ where a is the outcome "Apricot sold," and b is the outcome "Banana sold."

- The sample space in Example 1.3 is

$$S = \{aaa, aab, aba, abb, baa, bab, bba, bbb\} \tag{1.5}$$

- The sample space in Example 1.4 is $S = \{0, 1, 2, 3\}$.

Definition 1.3 Event

An event *is a set of outcomes of an experiment.*

- We say that *an event occurs* when a certain phenomenon is observed

Example 1.6

Observe the number of minutes a customer spends in the Phonesmart store. An outcome T is a nonnegative real number. The sample space is $S = \{T|T \geq 0\}$. The event "the customer stays longer than five minutes" is $\{T|T > 5\}$.

Quiz 1.1

Monitor three consecutive packets going through a Internet router. Based on the packet header, each packet can be classified as either video (v) if it was sent from a Youtube server or as ordinary data (d). Your observation is a sequence of three letters (each letter is either v or d). For example, two video packets followed by one data packet corresponds to vvd. Write the elements of the following sets:

$$A_1 = \{\text{second packet is video}\}, \qquad B_1 = \{\text{second packet is data}\},$$
$$A_2 = \{\text{all packets are the same}\}, \qquad B_2 = \{\text{video and data alternate}\},$$
$$A_3 = \{\text{one or more video packets}\}, \quad B_3 = \{\text{two or more data packets}\}.$$

For each pair of events A_1 and B_1, A_2 and B_2, and so on, identify whether the pair of events is either mutually exclusive or collectively exhaustive or both.

Set Theory	Probability Theory
Universal set	Sample space
Set	Event
Element	Outcome

Section 1.3

Probability Axioms

- In this section, we define ‘probability’
- Axioms are propositions that are taken as truth for granted without proofs
  - They are starting point for deducing other facts

Definition 1.4 Axioms of Probability

A probability measure $P[\cdot]$ *is a function that maps events in the sample space to real numbers such that*

Axiom 1 *For any event* A, $P[A] \geq 0$.

Axiom 2 $P[S] = 1$.

Axiom 3 *For any countable collection* $A_1, A_2, \ldots$ *of mutually exclusive events*

$$P[A_1 \cup A_2 \cup \cdots] = P[A_1] + P[A_2] + \cdots .$$

Theorem 1.1

For mutually exclusive events A_1 and A_2,

$$\mathsf{P}[A_1 \cup A_2] = \mathsf{P}[A_1] + \mathsf{P}[A_2].$$

Theorem 1.2

If $A = A_1 \cup A_2 \cup \cdots \cup A_m$ and $A_i \cap A_j = \varnothing$ for $i \neq j$, then

$$\mathsf{P}[A] = \sum_{i=1}^{m} \mathsf{P}[A_i].$$

Theorem 1.3

The probability measure $\mathsf{P}[\cdot]$ satisfies

(a) $\mathsf{P}[\varnothing] = 0$.

(b) $\mathsf{P}[A^c] = 1 - \mathsf{P}[A]$.

(c) For any A and B (not necessarily mutually exclusive),

$$\mathsf{P}\left[A \cup B\right] = \mathsf{P}\left[A\right] + \mathsf{P}\left[B\right] - \mathsf{P}\left[A \cap B\right].$$

(d) If $A \subset B$, then $\mathsf{P}[A] \leq \mathsf{P}[B]$.

Theorem 1.4

The probability of an event $B = \{s_1, s_2, \ldots, s_m\}$ is the sum of the probabilities of the outcomes contained in the event:

$$\mathsf{P}[B] = \sum_{i=1}^{m} \mathsf{P}[\{s_i\}].$$

Theorem 1.5

For an experiment with sample space $S = \{s_1, \ldots, s_n\}$ in which each outcome s_i is equally likely,

$$\mathsf{P}[s_i] = 1/n \qquad 1 \leq i \leq n.$$

Example 1.8 Problem

Roll a six-sided die in which all faces are equally likely. What is the probability of each outcome? Find the probabilities of the events: "Roll 4 or higher," "Roll an even number," and "Roll the square of an integer."

Quiz 1.2

A student's test score T is an integer between 0 and 100 corresponding to the experimental outcomes $s_0, \ldots, s_{100}$. A score of 90 to 100 is an A, 80 to 89 is a B, 70 to 79 is a C, 60 to 69 is a D, and below 60 is a failing grade of F. If all scores between 51 and 100 are equally likely and a score of 50 or less never occurs, find the following probabilities:

(a) $\text{P}[\{s_{100}\}]$

(b) $\text{P}[A]$

(c) $\text{P}[F]$

(d) $\text{P}[T < 90]$

(e) P[a C grade or better]

(f) P[student passes]