Section 1.3

Conditional Probability
Definition 1.5 Conditional Probability

The conditional probability of the event $A$ given the occurrence of the event $B$ is

$$P[A|B] = \frac{P[AB]}{P[B]}.$$
Example 1.9

Consider an experiment that consists of testing two integrated circuits (IC chips) that come from the same silicon wafer and observing in each case whether a chip is accepted (a) or rejected (r). The sample space of the experiment is $S = \{rr, ra, ar, aa\}$. Let $B$ denote the event that the first chip tested is rejected. Mathematically, $B = \{rr, ra\}$. Similarly, let $A = \{rr, ar\}$ denote the event that the second chip is a failure.

Example 1.10 Problem

With respect to Example 1.9, consider the a priori probability model

$$P[rr] = 0.01, \quad P[ra] = 0.01, \quad P[ar] = 0.01, \quad P[aa] = 0.97.$$  \hspace{1cm} (1.9)

Find the probability of $A = \text{“second chip rejected”}$ and $B = \text{“first chip reject}$.

Also find the conditional probability that the second chip is a reject given that the first chip is a reject.
Example 1.11 Problem

Roll two fair four-sided dice. Let $X_1$ and $X_2$ denote the number of dots that appear on die 1 and die 2, respectively. Let $A$ be the event $X_1 \geq 2$. What is $P[A]$? Let $B$ denote the event $X_2 > X_1$. What is $P[B]$? What is $P[A|B]$?
Theorem 1.6

A conditional probability measure $P[A|B]$ has the following properties that correspond to the axioms of probability.

Axiom 1: $P[A|B] \geq 0$.


Axiom 3: If $A = A_1 \cup A_2 \cup \cdots$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$P[A|B] = P[A_1|B] + P[A_2|B] + \cdots$$
Monitor three consecutive packets going through an Internet router. Classify each one as either video \( (v) \) or data \( (d) \). Your observation is a sequence of three letters (each one is either \( v \) or \( d \)). For example, three video packets corresponds to \( vvv \). The outcomes \( vvv \) and \( ddd \) each have probability 0.2 whereas each of the other outcomes \( vvd, vdv, vdd, dvv, dvd, \) and \( ddv \) has probability 0.1. Count the number of video packets \( N_V \) in the three packets you have observed. Describe in words and also calculate the following probabilities:

(a) \( P[N_V = 2] \)

(b) \( P[N_V \geq 1] \)

(c) \( P\{vvd\}|N_V = 2 \)

(d) \( P\{ddv\}|N_V = 2 \)

(e) \( P[N_V = 2|N_V \geq 1] \)

(f) \( P[N_V \geq 1|N_V = 2] \)
Section 1.4

Partitions and the Law of Total Probability
Example 1.12 Problem

Flip four coins, a penny, a nickel, a dime, and a quarter. Examine the coins in order (penny, then nickel, then dime, then quarter) and observe whether each coin shows a head ($h$) or a tail ($t$). What is the sample space? How many elements are in the sample space?
Example 1.12 Solution

The sample space consists of 16 four-letter words, with each letter either \( h \) or \( t \). For example, the outcome \( tthh \) refers to the penny and the nickel showing tails and the dime and quarter showing heads. There are 16 members of the sample space.
Continuing Example 1.12, let $B_i = \{\text{outcomes with } i \text{ heads}\}$. Each $B_i$ is an event containing one or more outcomes. For example,

$$B_1 = \{ttth, ttth, thtt, httt\}$$

contains four outcomes. The set

$$B = \{B_0, B_1, B_2, B_3, B_4\}$$

is a partition. Its members are mutually exclusive and collectively exhaustive. It is not a sample space because it lacks the finest-grain property. Learning that an experiment produces an event $B_1$ tells you that one coin came up heads, but it doesn’t tell you which coin it was.
Theorem 1.7

For a partition $B = \{B_1, B_2, \ldots\}$ and any event $A$ in the sample space, let $C_i = A \cap B_i$. For $i \neq j$, the events $C_i$ and $C_j$ are mutually exclusive and

$$A = C_1 \cup C_2 \cup \cdots.$$
Example 1.14

In the coin-tossing experiment of Example 1.12, let $A$ equal the set of outcomes with less than three heads:

$$A = \{tttt, httt, thtt, ttht, ttth, hhtt, htth, hthh, thhh, thth, thht\}. \quad (1.15)$$

From Example 1.13, let $B_i = \{\text{outcomes with } i \text{ heads}\}$. Since $\{B_0, \ldots, B_4\}$ is a partition, Theorem 1.7 states that

$$A = (A \cap B_0) \cup (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4) \quad (1.16)$$

In this example, $B_i \subset A$, for $i = 0, 1, 2$. Therefore $A \cap B_i = B_i$ for $i = 0, 1, 2$. Also, for $i = 3$ and $i = 4$, $A \cap B_i = \emptyset$ so that $A = B_0 \cup B_1 \cup B_2$, a union of mutually exclusive sets. In words, this example states that the event “less than three heads” is the union of events “zero heads,” “one head,” and “two heads.”
Theorem 1.8

For any event $A$, and partition $\{B_1, B_2, \ldots, B_m\}$,

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i].$$
Example 1.15

A company has a model of email use. It classifies all emails as either long (l), if they are over 10 MB in size, or brief (b). It also observes whether the email is just text (t), has attached images (i), or has an attached video (v). This model implies an experiment in which the procedure is to monitor an email and the observation consists of the type of email, t, i, or v, and the length, l or b. The sample space has six outcomes: \( S = \{lt, bt, li, bi, lv, bv\} \). In this problem, each email is classified in two ways: by length and by type. Using \( L \) for the event that an email is long and \( B \) for the event that a email is brief, \( \{L, B\} \) is a partition. Similarly, the text (T), image (I), and video (V) classification is a partition \( \{T, I, V\} \). The sample space can be represented by a table in which the rows and columns are labeled by events and the intersection of each row and column event contains a single outcome. The corresponding table entry is the probability of that outcome. In this case, the table is

<table>
<thead>
<tr>
<th></th>
<th>T</th>
<th>I</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L )</td>
<td>0.3</td>
<td>0.12</td>
<td>0.15</td>
</tr>
<tr>
<td>( B )</td>
<td>0.2</td>
<td>0.08</td>
<td>0.15</td>
</tr>
</tbody>
</table>

(1.17)

For example, from the table we can read that the probability of a brief image email is \( P[bi] = P[BI] = 0.08 \). Note that \( \{T, I, V\} \) is a partition corresponding to \( \{B_1, B_2, B_3\} \) in Theorem 1.8. Thus we can apply Theorem 1.8 to find the probability of a long email:

\[
\]

(1.18)
Theorem 1.9  Law of Total Probability

For a partition \( \{B_1, B_2, \ldots, B_m\} \) with \( P[B_i] > 0 \) for all \( i \),

\[
P[A] = \sum_{i=1}^{m} P[A|B_i] P[B_i].
\]
Example 1.16 Problem

A company has three machines $B_1$, $B_2$, and $B_3$ making 1 kΩ resistors. Resistors within 50 Ω of the nominal value are considered acceptable. It has been observed that 80% of the resistors produced by $B_1$ and 90% of the resistors produced by $B_2$ are acceptable. The percentage for machine $B_3$ is 60%. Each hour, machine $B_1$ produces 3000 resistors, $B_2$ produces 4000 resistors, and $B_3$ produces 3000 resistors. All of the resistors are mixed together at random in one bin and packed for shipment. What is the probability that the company ships an acceptable resistor?
Theorem 1.10  Bayes’ theorem

\[ P[B|A] = \frac{P[A|B] P[B]}{P[A]} . \]
Example 1.17 Problem

In Example 1.16 about a shipment of resistors from the factory, we learned that:

- The probability that a resistor is from machine $B_3$ is $P[B_3] = 0.3$.

- The probability that a resistor is acceptable — i.e., within 50 Ω of the nominal value — is $P[A] = 0.78$.

- Given that a resistor is from machine $B_3$, the conditional probability that it is acceptable is $P[A|B_3] = 0.6$.

What is the probability that an acceptable resistor comes from machine $B_3$?
Monitor customer behavior in the Phonesmart store. Classify the behavior as buying \((B)\) if a customer purchases a smartphone. Otherwise the behavior is no purchase \((N)\). Classify the time a customer is in the store as long \((L)\) if the customer stays more than three minutes; otherwise classify the amount of time as rapid \((R)\). Based on experience with many customers, we use the probability model \(P[N] = 0.7\), \(P[L] = 0.6\), \(P[N\cap L] = 0.35\). Find the following probabilities:

(a) \(P[B \cup L]\)
(b) \(P[N \cup L]\)
(c) \(P[N \cup B]\)
(d) \(P[L\cap R]\)
Section 1.5

Independence
Definition 1.6 Two Independent Events

Events $A$ and $B$ are independent if and only if

$$P[AB] = P[A]P[B].$$
Example 1.18 Problem

Suppose that for the experiment monitoring three purchasing decisions in Example 1.7, each outcome (a sequence of three decisions, each either buy or not buy) is equally likely. Are the events $B_2$ that the second customer purchases a phone and $N_2$ that the second customer does not purchase a phone independent? Are the events $B_1$ and $B_2$ independent?
Integrated circuits undergo two tests. A mechanical test determines whether pins have the correct spacing, and an electrical test checks the relationship of outputs to inputs. We assume that electrical failures and mechanical failures occur independently. Our information about circuit production tells us that mechanical failures occur with probability 0.05 and electrical failures occur with probability 0.2. What is the probability model of an experiment that consists of testing an integrated circuit and observing the results of the mechanical and electrical tests?
Definition 1.7 Three Independent Events

A_1, A_2, and A_3 are mutually independent if and only if

(a) A_1 and A_2 are independent,
(b) A_2 and A_3 are independent,
(c) A_1 and A_3 are independent,
(d) P[A_1 \cap A_2 \cap A_3] = P[A_1] P[A_2] P[A_3].
Example 1.20 Problem

In an experiment with equiprobable outcomes, the partition is \( S = \{1, 2, 3, 4\} \). \( P[s] = \frac{1}{4} \) for all \( s \in S \). Are the events \( A_1 = \{1, 3, 4\} \), \( A_2 = \{2, 3, 4\} \), and \( A_3 = \emptyset \) mutually independent?
More than Two Independent

**Definition 1.8 Events**

If $n \geq 3$, the events $A_1, A_2, \ldots, A_n$ are mutually independent if and only if

(a) all collections of $n - 1$ events chosen from $A_1, A_2, \ldots A_n$ are mutually independent,

(b) $P[A_1 \cap A_2 \cap \cdots \cap A_n] = P[A_1] P[A_2] \cdots P[A_n]$. 
Quiz 1.5

Monitor two consecutive packets going through a router. Classify each one as video \((v)\) if it was sent from a Youtube server or as ordinary data \((d)\) otherwise. Your observation is a sequence of two letters (either \(v\) or \(d\)). For example, two video packets corresponds to \(vv\). The two packets are independent and the probability that any one of them is a video packet is 0.8. Denote the identity of packet \(i\) by \(C_i\). If packet \(i\) is a video packet, then \(C_i = v\); otherwise, \(C_i = d\). Count the number \(N_V\) of video packets in the two packets you have observed. Determine whether the following pairs of events are independent:

(a) \(\{N_V = 2\}\) and \(\{N_V \geq 1\}\)

(b) \(\{N_V \geq 1\}\) and \(\{C_1 = v\}\)

(c) \(\{C_2 = v\}\) and \(\{C_1 = d\}\)

(d) \(\{C_2 = v\}\) and \(\{N_V\) is even\)