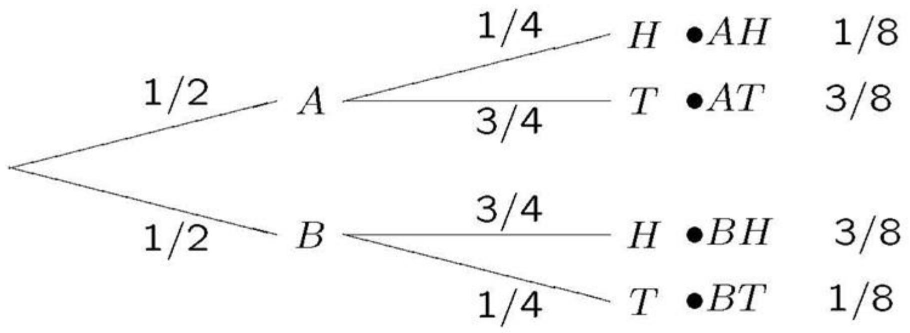
Problem 2.1.4 Solution

The tree for this experiment is



The probability that you guess correctly is

$$P[C] = P[AT] + P[BH] = 3/8 + 3/8 = 3/4. \quad (1)$$

Problem 2.1.6 Solution

The $\mathsf{P}[-|H]$ is the probability that a person who has HIV tests negative for the disease. This is referred to as a false-negative result. The case where a person who does not have HIV but tests positive for the disease, is called a false-positive result and has probability $\mathsf{P}[+|H^c]$. Since the test is correct 99% of the time,

$$\mathsf{P}[-|H] = \mathsf{P}[+|H^c] = 0.01. \tag{1}$$

Now the probability that a person who has tested positive for HIV actually has the disease is

$$\mathsf{P}[H|+] = \frac{\mathsf{P}[+,H]}{\mathsf{P}[+]} = \frac{\mathsf{P}[+,H]}{\mathsf{P}[+,H] + \mathsf{P}[+,H^c]}. \tag{2}$$

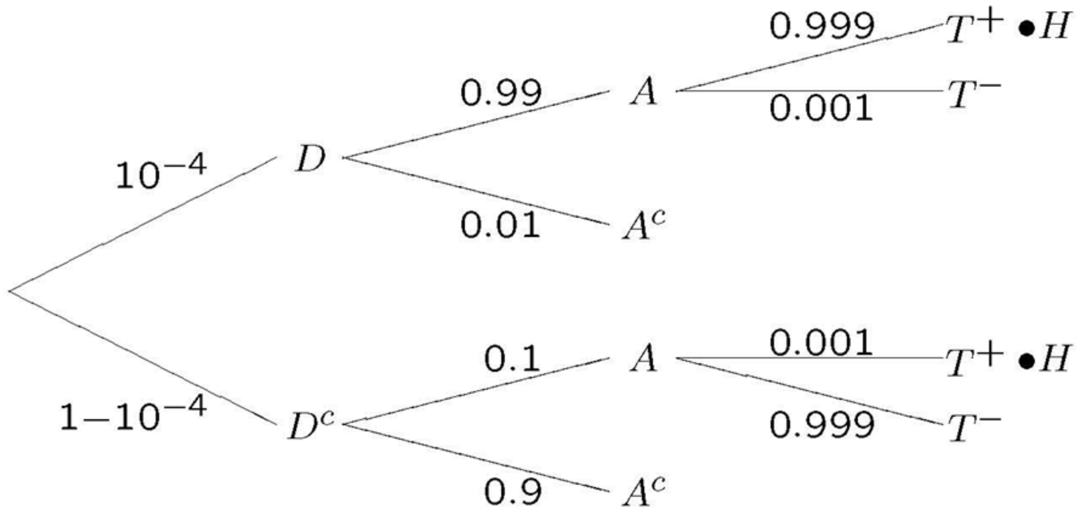
We can use Bayes' formula to evaluate these joint probabilities.

$$\begin{aligned} \mathsf{P}[H|+] &= \frac{\mathsf{P}[+|H]\,\mathsf{P}[H]}{\mathsf{P}[+|H]\,\mathsf{P}[H] + \mathsf{P}[+|H^c]\,\mathsf{P}[H^c]} \\ &= \frac{(0.99)(0.0002)}{(0.99)(0.0002) + (0.01)(0.9998)} \\ &= 0.0194. \end{aligned} \tag{3}$$

Thus, even though the test is correct 99% of the time, the probability that a random person who tests positive actually has HIV is less than 0.02. The reason this probability is so low is that the a priori probability that a person has HIV is very small.

Problem 2.1.9 Solution

We start with a tree diagram:



[Continued]

Problem 2.1.9 Solution (Continued 2)

(a) Here we are asked to calculate the conditional probability $P[D|A]$. In this part, its simpler to ignore the last branches of the tree that indicate the lab test result. This yields

$$\begin{aligned} P[D|A] = \frac{P[DA]}{P[A]} &= \frac{P[AD]}{P[DA] + P[D^cA]} \\ &= \frac{(10^{-4})(0.99)}{(10^{-4})(0.99) + (0.1)(1 - 10^{-4})} \\ &= 9.89 \times 10^{-4}. \end{aligned} \quad (1)$$

The probability of the defect D given the arrhythmia A is still quite low because the probability of the defect is so small.

(b) Since the heart surgery occurs if and only if the event T^+ occurs, H and T^+ are the same event and (from the previous part)

$$\begin{aligned} P[H|D] = P\left[T^+|D\right] &= \frac{P\left[DT^+\right]}{P[D]} \\ &= \frac{10^{-4}(0.99)(0.999)}{10^{-4}} = (0.99)(0.999). \end{aligned} \quad (2)$$

[Continued]

Problem 2.1.9 Solution (Continued 3)

(c) Since the heart surgery occurs if and only if the event T^+ occurs, H and T^+ are the same event and (from the previous part)

$$\begin{aligned} \mathsf{P}[H|D^c] = \mathsf{P}\left[T^+|D^c\right] &= \frac{\mathsf{P}\left[D^cT^+\right]}{\mathsf{P}[D^c]} \\ &= \frac{(1-10^{-4})(0.1)(0.001)}{1-10^{-4}} = 10^{-4}. \end{aligned} \tag{3}$$

(d) Heart surgery occurs with probability

$$\begin{aligned} \mathsf{P}[H] &= \mathsf{P}[H|D]\,\mathsf{P}[D] + \mathsf{P}[H|D^c]\,\mathsf{P}[D^c] \\ &= (0.99)(0.999)(10^{-4}) + (10^{-4})(1-10^{-4}) \\ &= 1.99 \times 10^{-4}. \end{aligned} \tag{4}$$

[Continued]

Problem 2.1.9 Solution (Continued 4)

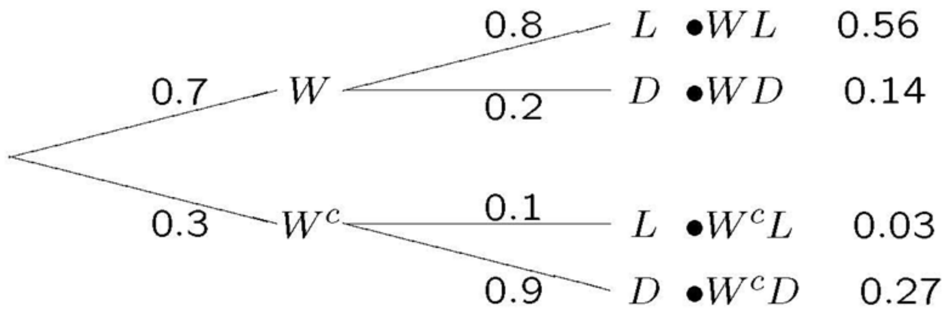
(e) Given that heart surgery was performed, the probability the child had no defect is

$$P[D^c|H] = \frac{P[D^cH]}{P[H]}$$
$$= \frac{(1-10^{-4})(0.1)(0.001)}{(0.99)(0.999)(10^{-4}) + (10^{-4})(1-10^{-4})}$$
$$= \frac{1-10^{-4}}{2-10^{-2}-10^{-3}+10^{-4}} = 0.5027. \quad (5)$$

Because the arrythmia is fairly common and the lab test is not fully reliable, roughly half of all the heart surgeries are performed on healthy infants.

Problem 2.1.12 Solution

The starting point is to draw a tree of the experiment. We define the events W that the plant is watered, L that the plant lives, and D that the plant dies. The tree diagram is



[Continued]

Problem 2.1.12 Solution (Continued 2)

It follows that

(a) $P[L] = P[WL] + P[W^cL] = 0.56 + 0.03 = 0.59.$

(b)

$$P\left[W^c|D\right] = \frac{P\left[W^cD\right]}{P\left[D\right]} = \frac{0.27}{0.14 + 0.27} = \frac{27}{41}. \qquad (1)$$

(c) $P[D|W^c] = 0.9.$

In informal conversation, it can be confusing to distinguish between $P[D|W^c]$ and $P[W^c|D]$; however, they are simple once you draw the tree.

Problem 2.2.11 Solution

(a) We can find the number of valid starting lineups by noticing that the swingman presents three situations: (1) the swingman plays guard, (2) the swingman plays forward, and (3) the swingman doesn't play. The first situation is when the swingman can be chosen to play the guard position, and the second where the swingman can only be chosen to play the forward position. Let N_i denote the number of lineups corresponding to case i. Then we can write the total number of lineups as $N_1 + N_2 + N_3$. In the first situation, we have to choose 1 out of 3 centers, 2 out of 4 forwards, and 1 out of 4 guards so that

$$N_1 = \binom{3}{1}\binom{4}{2}\binom{4}{1} = 72. \tag{1}$$

In the second case, we need to choose 1 out of 3 centers, 1 out of 4 forwards and 2 out of 4 guards, yielding

$$N_2 = \binom{3}{1}\binom{4}{1}\binom{4}{2} = 72. \tag{2}$$

Finally, with the swingman on the bench, we choose 1 out of 3 centers, 2 out of 4 forward, and 2 out of four guards. This implies

$$N_3 = \binom{3}{1}\binom{4}{2}\binom{4}{2} = 108, \tag{3}$$

and the total number of lineups is $N_1 + N_2 + N_3 = 252$.

Problem 2.2.12 Solution

What our design must specify is the number of boxes on the ticket, and the number of specially marked boxes. Suppose each ticket has n boxes and $5+k$ specially marked boxes. Note that when $k>0$, a winning ticket will still have k unscratched boxes with the special mark. A ticket is a winner if each time a box is scratched off, the box has the special mark. Assuming the boxes are scratched off randomly, the first box scratched off has the mark with probability $(5+k)/n$ since there are $5+k$ marked boxes out of n boxes. Moreover, if the first scratched box has the mark, then there are $4+k$ marked boxes out of $n-1$ remaining boxes. Continuing this argument, the probability that a ticket is a winner is

$$p = \frac{5+k}{n}\frac{4+k}{n-1}\frac{3+k}{n-2}\frac{2+k}{n-3}\frac{1+k}{n-4} = \frac{(k+5)!(n-5)!}{k!n!}. \tag{1}$$

By careful choice of n and k, we can choose p close to 0.01. For example,

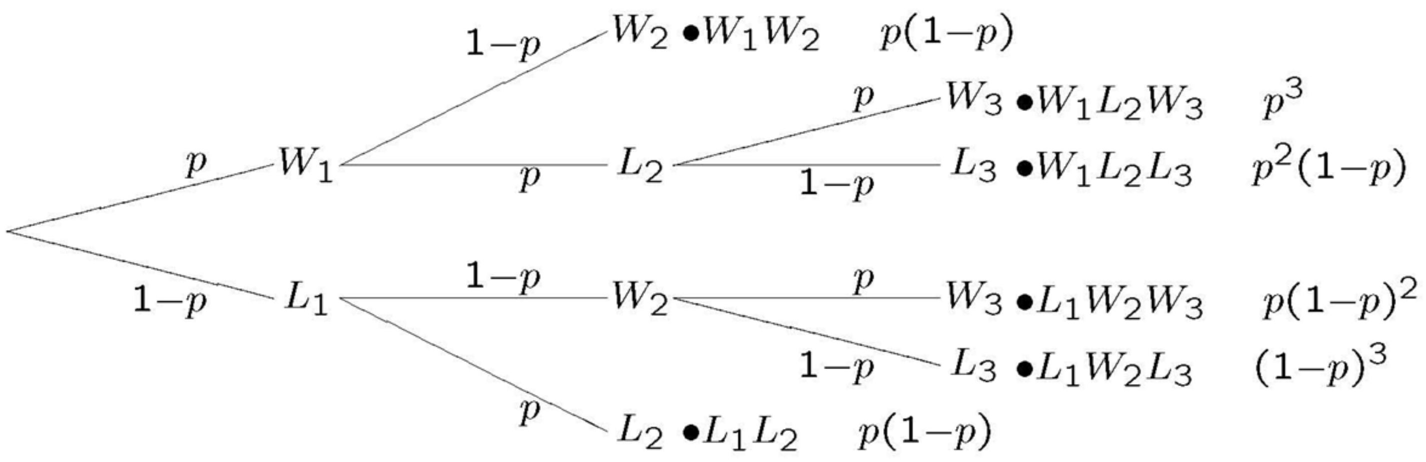
n	9	11	14	17
k	0	1	2	3
p	0.0079	0.012	0.0105	0.0090

(2)

A gamecard with $N=14$ boxes and $5+k=7$ shaded boxes would be quite reasonable.

Problem 2.3.4 Solution

For the team with the homecourt advantage, let W_i and L_i denote whether game i was a win or a loss. Because games 1 and 3 are home games and game 2 is an away game, the tree is



[Continued]

Problem 2.3.4 Solution (Continued 2)

The probability that the team with the home court advantage wins is

$$\begin{aligned} P[H] &= P[W_1W_2] + P[W_1L_2W_3] + P[L_1W_2W_3] \\ &= p(1-p) + p^3 + p(1-p)^2. \end{aligned} \quad (1)$$

Note that $P[H] \leq p$ for $1/2 \leq p \leq 1$. Since the team with the home court advantage would win a 1 game playoff with probability p, the home court team is less likely to win a three game series than a 1 game playoff!

Problem 2.3.5 Solution

(a) There are 3 group 1 kickers and 6 group 2 kickers. Using G_i to denote that a group i kicker was chosen, we have

$$P[G_1] = 1/3, \qquad P[G_2] = 2/3. \tag{1}$$

In addition, the problem statement tells us that

$$P[K|G_1] = 1/2, \qquad P[K|G_2] = 1/3. \tag{2}$$

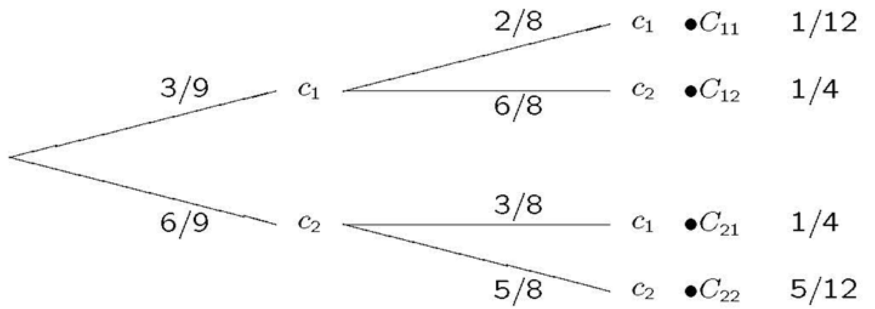
Combining these facts using the Law of Total Probability yields

$$\begin{aligned} P[K] &= P[K|G_1] P[G_1] + P[K|G_2] P[G_2] \\ &= (1/2)(1/3) + (1/3)(2/3) = 7/18. \end{aligned} \tag{3}$$

(b) To solve this part, we need to identify the groups from which the first and second kicker were chosen. Let c_i indicate whether a kicker was chosen from group i and let C_{ij} indicate that the first kicker was chosen from group i and the second kicker from group j. The experiment to choose the kickers is described by the sample tree:

[Continued]

Problem 2.3.5 Solution (Continued 2)



Since a kicker from group 1 makes a kick with probability 1/2 while a kicker from group 2 makes a kick with probability 1/3,

$$P[K_1K_2|C_{11}] = (1/2)^2, \qquad P[K_1K_2|C_{12}] = (1/2)(1/3), \tag{4}$$

$$P[K_1K_2|C_{21}] = (1/3)(1/2), \qquad P[K_1K_2|C_{22}] = (1/3)^2. \tag{5}$$

By the law of total probability,

$$\begin{aligned} P[K_1K_2] &= P[K_1K_2|C_{11}]\,P[C_{11}] + P[K_1K_2|C_{12}]\,P[C_{12}] \\ &\quad + P[K_1K_2|C_{21}]\,P[C_{21}] + P[K_1K_2|C_{22}]\,P[C_{22}] \\ &= \frac{1}{4}\frac{1}{12} + \frac{1}{6}\frac{1}{4} + \frac{1}{6}\frac{1}{4} + \frac{1}{9}\frac{5}{12} = 15/96. \end{aligned} \tag{6}$$

[Continued]

Problem 2.3.5 Solution (Continued 3)

It should be apparent that $P[K_1] = P[K]$ from part (a). Symmetry should also make it clear that $P[K_1] = P[K_2]$ since for any ordering of two kickers, the reverse ordering is equally likely. If this is not clear, we derive this result by calculating $P[K_2|C_{ij}]$ and using the law of total probability to calculate $P[K_2]$.

$$P[K_2|C_{11}] = 1/2, \qquad P[K_2|C_{12}] = 1/3, \tag{7}$$

$$P[K_2|C_{21}] = 1/2, \qquad P[K_2|C_{22}] = 1/3. \tag{8}$$

By the law of total probability,

$$\begin{aligned} P[K_2] &= P[K_2|C_{11}]\,P[C_{11}] + P[K_2|C_{12}]\,P[C_{12}] \\ &\quad + P[K_2|C_{21}]\,P[C_{21}] + P[K_2|C_{22}]\,P[C_{22}] \\ &= \frac{1}{2}\frac{1}{12} + \frac{1}{3}\frac{1}{4} + \frac{1}{2}\frac{1}{4} + \frac{1}{3}\frac{5}{12} = \frac{7}{18}. \end{aligned} \tag{9}$$

We observe that K_1 and K_2 are not independent since

$$P[K_1K_2] = \frac{15}{96} \neq \left(\frac{7}{18}\right)^2 = P[K_1]\,P[K_2]. \tag{10}$$

Note that $15/96$ and $(7/18)^2$ are close but not exactly the same. The reason K_1 and K_2 are dependent is that if the first kicker is successful, then it is more likely that kicker is from group 1. This makes it more likely that the second kicker is from group 2 and is thus more likely to miss. [Continued]

Problem 2.3.5 Solution (Continued 4)

(c) Once a kicker is chosen, each of the 10 field goals is an independent trial. If the kicker is from group 1, then the success probability is 1/2. If the kicker is from group 2, the success probability is 1/3. Out of 10 kicks, there are 5 misses iff there are 5 successful kicks. Given the type of kicker chosen, the probability of 5 misses is

$$\mathsf{P}[M|G_1] = \binom{10}{5}(1/2)^5(1/2)^5, \qquad \mathsf{P}[M|G_2] = \binom{10}{5}(1/3)^5(2/3)^5. \tag{11}$$

We use the Law of Total Probability to find

$$\begin{aligned}\mathsf{P}[M] &= \mathsf{P}[M|G_1]\,\mathsf{P}[G_1] + \mathsf{P}[M|G_2]\,\mathsf{P}[G_2] \\ &= \binom{10}{5}\left((1/3)(1/2)^{10} + (2/3)(1/3)^5(2/3)^5\right).\end{aligned} \tag{12}$$