Chapter 6

Probability Models of Derived Random Variables

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Warming Up

Theorem 6.1

For discrete random variables X and Y, the derived random variable $W = g(X, Y)$ has PMF

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y).$$

Example 6.1 Problem

$P_{L,X}(l,x)$	$x=40$	$x=60$
$l=1$	0.15	0.1
$l=2$	0.3	0.2
$l=3$	0.15	0.1

A firm sends out two kinds of newsletters. One kind contains only text and grayscale images and requires 40 cents to print each page. The other kind contains color pictures that cost 60 cents per page. Newsletters can be 1, 2, or 3 pages long. Let the random variable L represent the length of a newsletter in pages. $S_L = \{1,2,3\}$. Let the random variable X represent the cost in cents to print each page. $S_X = \{40,60\}$. After observing many newsletters, the firm has derived the probability model shown above. Let $W = g(L,X) = LX$ be the total cost in cents of a newsletter. Find the range S_W and the PMF $P_W(w)$.

Section 6.2

$$\boxed{Y = g(X)}$$

Better Theorem

A random variable X has a pdf $f_X(x)$. Then, the pdf of a derived random variable $Y = g(X)$ is given by

$$f_Y(y) = \sum_{x_i : g(x_i) = y} \frac{f_X(x_i)}{|g'(x_i)|}$$

Theorem 6.2

If $W = aX$, where $a > 0$, then W has CDF and PDF

$$F_W(w) = F_X(w/a), \qquad f_W(w) = \frac{1}{a} f_X(w/a).$$

Example 6.3 Problem

The triangular PDF of X is

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.7}$$

Find the PDF of $W = aX$. Sketch the PDF of W for $a = 1/2, 1, 2$.

Theorem 6.4

If $W = X + b$,

$$F_W(w) = F_X(w - b), \qquad f_W(w) = f_X(w - b).$$

Example 6.4 Problem

Suppose X is the continuous uniform (-1, 3) random variable and $W = X^2$. Find the CDF $F_W(w)$ and PDF $f_W(w)$.

Theorem 6.5

Let U be a uniform $(0,1)$ random variable and let $F(x)$ denote a cumulative distribution function with an inverse $F^{-1}(u)$ defined for $0 < u < 1$. The random variable $X = F^{-1}(U)$ has CDF $F_X(x) = F(x)$.

Example 6.5 Problem

U is the uniform $(0,1)$ random variable and $X = g(U)$. Derive $g(U)$ such that X is the exponential (1) random variable.

Example 6.6 Problem

For a uniform $(0,1)$ random variable U, find a function $g(\cdot)$ such that $X = g(U)$ has a uniform (a,b) distribution.

Quiz 6.2

X is an exponential (λ) PDF. Show that $Y = \sqrt{X}$ is a Rayleigh random variable (see Appendix A.2). Express the Rayleigh parameter a in terms of the exponential parameter λ.

Rayleigh (a)

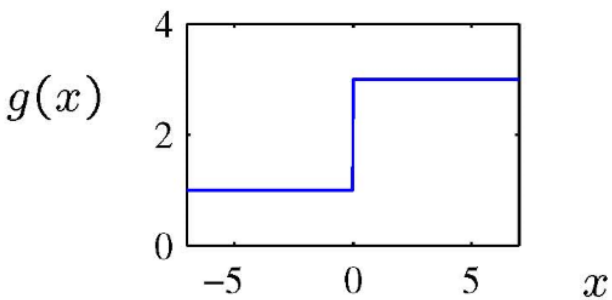
$f_X(x) = a^2 x \exp(-\frac{a^2 x^2}{2})$ for $x > 0$

$$Y = g(X)$$

with singularity

Example 6.7 Problem

Let X be a random variable with CDF $F_X(x)$. Let Y be the output of a clipping circuit, also referred to as a hard limiter, with the characteristic $Y = g(X)$ where

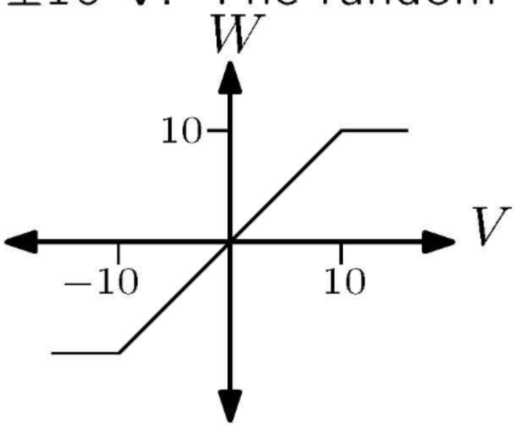


$$g(x) = \begin{cases} 1 & x \leq 0, \\ 3 & x > 0. \end{cases} \tag{6.21}$$

Express $F_Y(y)$ and $f_Y(y)$ in terms of $F_X(x)$ and $f_X(x)$.

Example 6.8 Problem

The output voltage of a microphone is a Gaussian random variable V with expected value $\mu_V = 0$ and standard deviation $\sigma_V = 5$ V. The microphone signal is the input to a soft limiter circuit with cutoff value ± 10 V. The random variable W is the output of the limiter:



$$W = g(V) = \begin{cases} -10 & V < -10, \\ V & -10 \le V \le 10, \\ 10 & V > 10. \end{cases} \tag{6.24}$$

What are the CDF and PDF of W?

Quiz 6.3

Random variable X is passed to a hard limiter that outputs Y. The PDF of X and the limiter output Y are

$$f_X(x) = \begin{cases} 1 - x/2 & 0 \leq x \leq 2, \\ 0 & \text{otherwise}, \end{cases} \qquad Y = \begin{cases} X & X \leq 1, \\ 1 & X > 1. \end{cases} \tag{6.28}$$

(a) What is the CDF $F_X(x)$?

(b) What is $\mathrm{P}[Y = 1]$?

(c) What is $F_Y(y)$?

(d) What is $f_Y(y)$?

Section 6.4

$$W = g(X, Y)$$

CDF first, PDF later

Theorem 6.6

For continuous random variables X and Y, the CDF of $W = g(X,Y)$ is

$$F_W(w) = \mathrm{P}\left[W \leq w\right] = \iint_{g(x,y) \leq w} f_{X,Y}(x,y)\; dx\, dy.$$

Theorem 6.7

For continuous random variables X and Y, the CDF of $W = \max(X,Y)$ is

$$F_W(w) = F_{X,Y}(w,w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x,y)\; dx\, dy.$$

Example 6.9 Problem

In Examples 5.7 and 5.9, X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6.29}$$

Find the PDF of $W = \max(X,Y)$.

Example 6.10 Problem

X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x+\mu y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.34}$$

Find the PDF of $W = Y/X$.

Quiz 6.4(B)

Find the CDF and the PDF of $W = XY$ when random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1, 0 \le y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.39}$$

Section 6.5

$$W = X + Y$$

Theorem 6.8

The PDF of $W = X + Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w - x)\, dx = \int_{-\infty}^{\infty} f_{X,Y}(w - y, y)\, dy.$$

Theorem 6.9

Convolution

When X and Y are independent random variables, the PDF of $W = X+Y$ is

$$f_W(w) = \int_{-\infty}^{\infty} f_X(w - y)\, f_Y(y)\, dy = \int_{-\infty}^{\infty} f_X(x)\, f_Y(w - x)\, dx.$$

Example 6.11 Problem

Find the PDF of $W = X + Y$ when X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le 1, 0 \le x \le 1, x + y \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.43}$$

Quiz 6.5

Let X and Y be independent exponential random variables with expected values $\mathsf{E}[X] = 1/3$ and $\mathsf{E}[Y] = 1/2$. Find the PDF of $W = X + Y$.