Chapter 14. Complex Integration

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The contents herein are based on the book "Advanced Engineering Mathematics" by E. Kreyszig and only for the course KEEE202, Korea University.

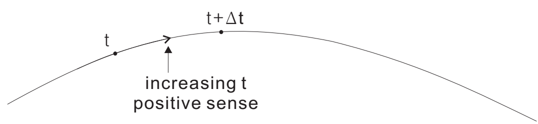
I. Line Integral

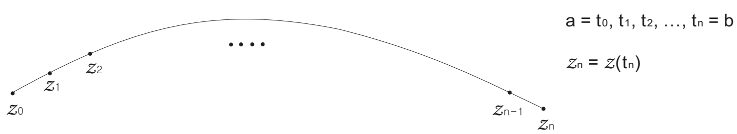
⋆ Line integral of f along a curve C

$$\int_C f(z)dz$$

The curve is given by

$$C: \quad z(t) = x(t) + iy(t) \qquad a \le t \le b$$





The line integral is given by the limit of the summation

$$\begin{aligned} S_n &= \sum_{m=1}^{n} f(z_{m-1})\Delta z_m \\ &= \sum_{m=1}^{n} f(z_{m-1})(z_m - z_{m-1}). \end{aligned}$$

In other words,

$$\begin{aligned} \int_C f(z)dz &\triangleq \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \sum_{m=1}^{n} f(z_{m-1})(z_m - z_{m-1}). \end{aligned}$$

Also, let $f(z) = u(z) + iv(z)$. Then, we have

$$\begin{aligned} S_n &= \sum_m \Big(u(z_{m-1}) + iv(z_{m-1})\Big)(\Delta x_m + i\Delta y_m) \\ &= \sum_m u\Delta x_m - \sum_m v\Delta y_m + i\Big[\sum_m u\Delta y_m + \sum_m v\Delta x_m\Big] \end{aligned}$$

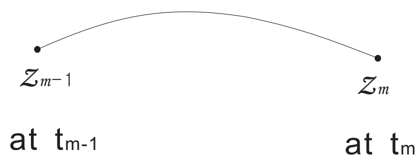
and

$$\int_C f(z)dz = \int_C udx - \int_C vdy + i\Big[\int_C udy + \int_C vdx\Big].$$

• Evaluating line integrals: method 1

$$\int_C f(z)dz = \int_a^b f[z(t)]\dot{z}(t)dt.$$

This is because



$$\begin{aligned}\Delta z_m &= z_m - z_{m-1} \\ &\cong \dot{z}(t_{m-1})(t_m - t_{m-1}) \\ &= \dot{z}(t_{m-1})\Delta t_m\end{aligned}$$

Then,

$$\begin{aligned}S_n &= \sum_{m=1}^{n} f(z_{m-1})\dot{z}(t_{m-1})\Delta t_m \\ &= \sum_{m=1}^{n} f[z(t_{m-1})]\dot{z}(t_{m-1})\Delta t_m\end{aligned}$$

and

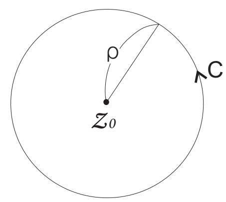
$$\lim_{n\to\infty} S_n = \int_a^b f[z(t)]\dot{z}(t)dt.$$

⋆ Ex 1) Let C be the unit circle, which has the counterclockwise orientation.

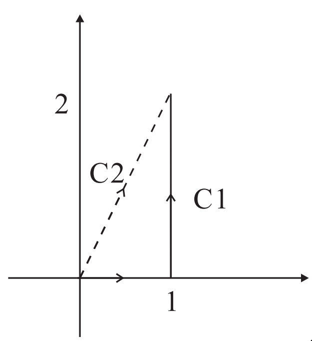
$$\oint_C \frac{1}{z}dz \ \ ?$$

⋆ Ex 2)

$$
\begin{aligned}
f(z) &= (z - z_0)^m \\
C: \quad z(t) &= z_0 + \rho e^{it} \qquad 0 \le t \le 2\pi
\end{aligned}
$$



⋆ Ex 3)



$f(z) = \text{Re } z$

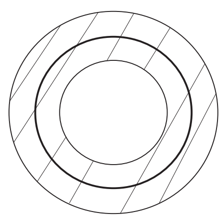
• Evaluating line integrals: method 2

⋆ Theorem

Suppose that $f(z)$ is analytic in a simply connected domain D. Then, there exists an indefinite integral of $f(z)$ in D, *i.e.*, there exists $F(z)$ such that $F'(z) = f(z)$. Also, for all paths in D joining two points z_0 and z_1

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

Note that D is called a simply connected domain, if every closed curve without self intersections encloses only points of D.



not simply connected

⋆ Ex 1)

$$\int_0^{1+i} z^2 dz = \frac{1}{3} z^3 \Big|_0^{1+i} = \frac{1}{3}(1+i)^3$$

⋆ Ex 2)

$$\begin{aligned} \int_{-i}^{i} \frac{1}{z} dz &= \operatorname{Ln} i - \operatorname{Ln}(-i) \\ &= i\frac{\pi}{2} - i(-\frac{\pi}{2}) \\ &= i\pi \end{aligned}$$

• ML-inequality

Let $|f(z)| \le M$ on C and L denote the length of C. Then,

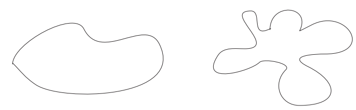
$$\Big| \int_C f(z)dz \Big| \le ML.$$

II. Cauchy's Integral Theorem

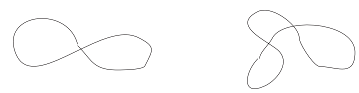
⋆ Theorem: If $f(z)$ is analytic in a simply connected domain D, then for every simple closed path C in D

$$\oint_C f(z)dz = 0$$

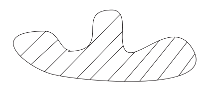
- Simple closed path



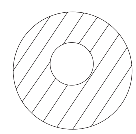
- Not simple closed path



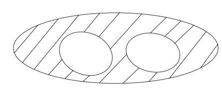
- Simply connected domain



- Doubly connected domain



- Triply connected domain



⋆ Examples

- Entire function

$$\oint_C e^z dz = \oint_C \cos z dz = \oint_C z^n dz = 0 \qquad n = 0, 1, 2, \cdots$$

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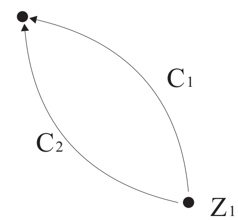
$$\oint_C \sec z dz = \oint_C \frac{1}{z^2+4} dz = 0 \qquad C: \text{ unit circle}$$

-

$$\oint_C \frac{1}{z^2} dz = 0$$

The last equality does not come from Cauchy's theorem.

⋆ Theorem (Path Independence): If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D, *i.e.* every path in D from z_1 to z_2 gives the same value of the integral.



$$\int_{c_1+(-c_2)} f(z)dz = \int_{c_1} f(z)dz + \int_{-c_2} f(z)dz = \int_{c_1} f(z)dz - \int_{c_2} f(z)dz. = 0$$

⋆ Principle of Deformation of Path:
We can deform the path of an integral, keeping the ends fixed, without causing a change in the integral value, as long as the deforming path contains only point at which f(z) is analytic.

For example, we can show that

$$\oint (z-z_0)^m dz = \begin{cases} 2\pi \mathrm{i} & \text{if } m=-1 \\ 0 & \text{if } m \neq -1 \text{ and an integer} \end{cases}$$

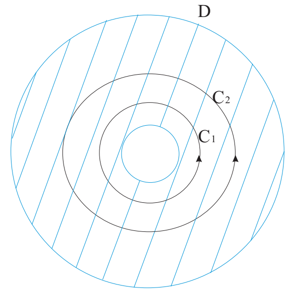
for any simple closed counterclockwise curve, containing Z_0 in its interior. Note that $f(z) = (z-z_0)^m$ is not analytic at $z = z_0$ when m is negative. However, the principle still holds true.

⋆ Existence of Indefinite Integral: If $f(z)$ is analytic in a simply connected domain D, then there exists $F(z)$ such that $F'(z) = f(z)$. And for all path from z_1 to z_2.

$$\int_c f(z)dz = F(z_2) - F(z_1)$$

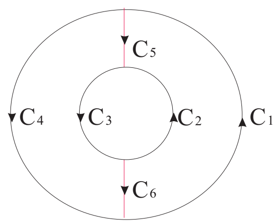
Sketch of proof)

⋆ Integral Theorem for multiply connected domains:



$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

Recall the principle of continuous deformation or note the following cutting argument.



$$\int_{c_1+c_5-c_2+c_6} + \int_{-c_3-c_5+c_4-c_6} = \int_{c_1+c_4} - \int_{c_2+c_3} = 0.$$

III. Cauchy's Integral Formula

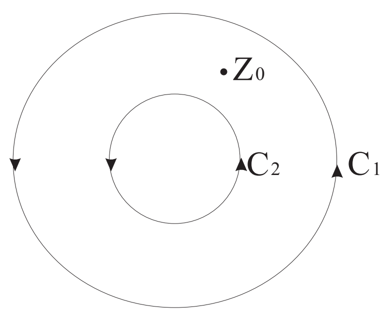
⋆ Theorem: Let D be a simply connected domain and $f(z)$ be analytic in D. Then, for any z_0 and for any simple closed path in D that encloses z_0, we have

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

⋆ Ex) $\oint_C \frac{z^3-6}{2z-i} dz$, where $i/2$ is inside C.

Also, note that for the following multiply connected domain, we have

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)}{z - z_0} dz + \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)}{z - z_0} dz.$$



IV. Complex analytic functions have derivatives of all orders

⋆ Recall

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)}dz$$

Also, we have

$$\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz \\ f''(z_0) &= \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz \\ &\vdots \\ f^{(n)}(z_0) &= \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz \end{aligned}$$

⋆ Ex 1)

$$\oint_C \frac{\cos z}{(z-\pi i)^2}dz$$

• Cauchy's Inequality:

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}$$

where $|f(z)| \leq M$ on the circle with radius r and center z_0.

• Liouville's Theorem: If an entire function is bounded, it is a constant function.