Signals and Systems

Continuous-Time Fourier Transform

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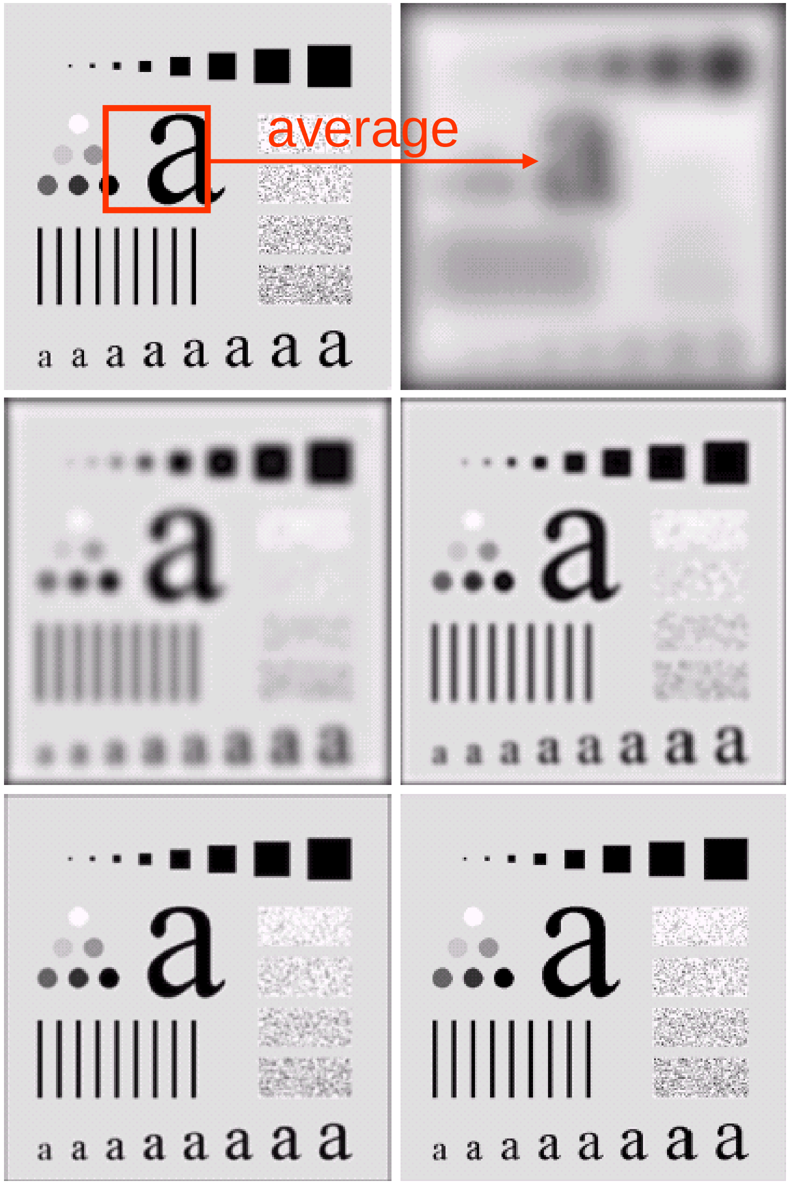
	continuous time	discrete time
periodic (series)	CTFS	DTFS
aperiodic (transform)	**CTFT**	DTFT

Lowpass Filtering – Blurring or Smoothing

Original



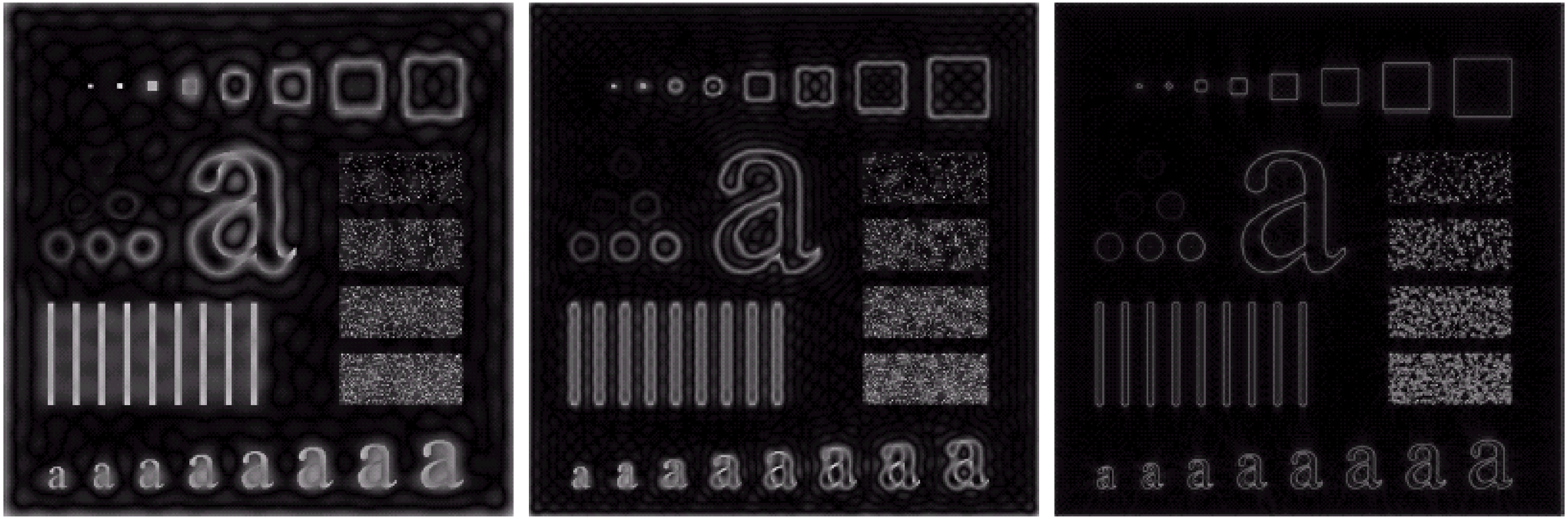
Strong LPF

e.g. 21x21 moving average filter

Less strong LPF

e.g. 11x11 moving average filter

Highpass Filtering – Edge Extraction



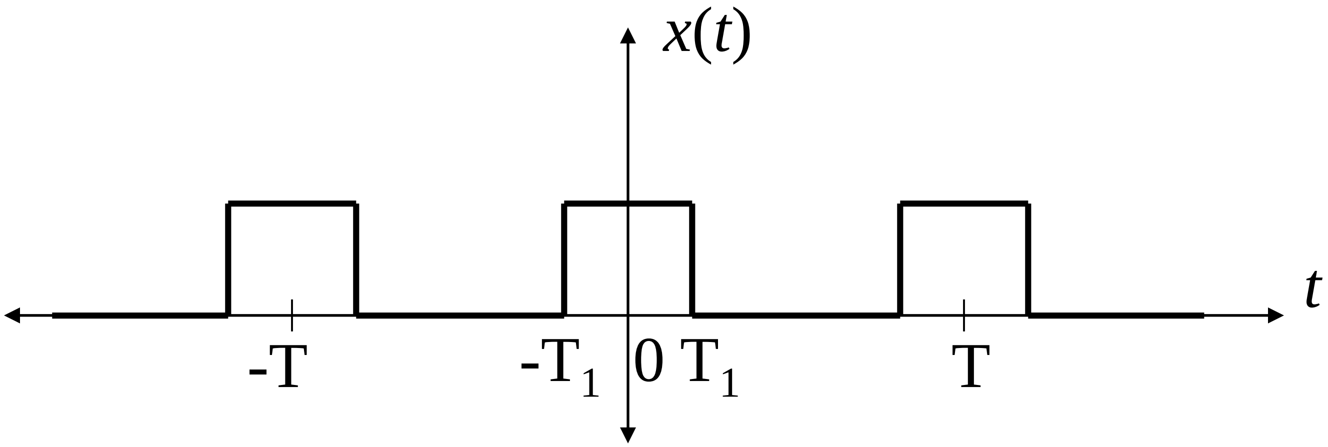
a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

CTFT Formula and Its Derivation

Bridge Between Fourier Series and Transform

- Consider the periodic signal x(t)

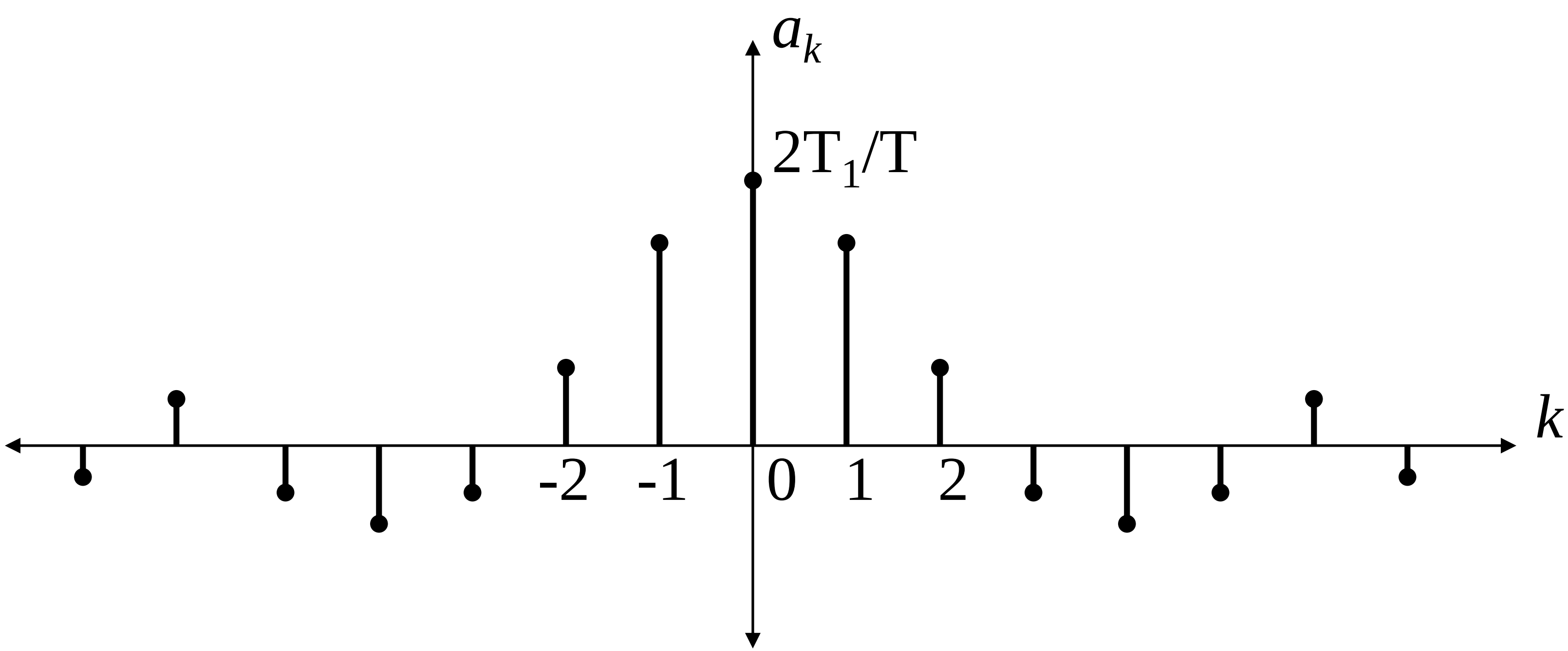


- Its Fourier coefficients are

$$a_k = \begin{cases} \dfrac{2T_1}{T}, & k = 0 \\ \dfrac{\sin(k\omega_0 T_1)}{k\pi} & k \neq 0 \end{cases}$$

Bridge Between Fourier Series and Transform

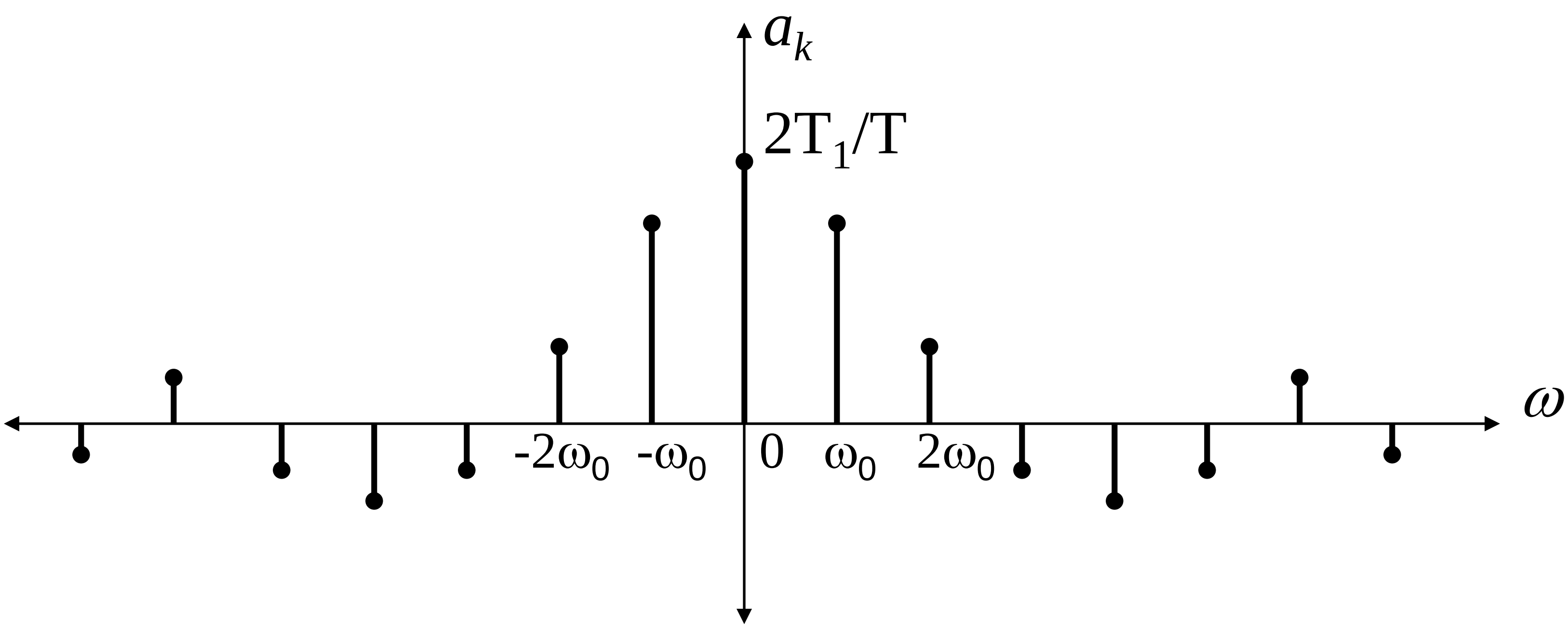
- Sketch a_k on the k-axis



- The sketch is obtained by sampling the sinc function.
- For each value of k, the signal x(t) has a periodic component with weight a_k. So, the above sketch shows the frequency content of the signal x(t).

Bridge Between Fourier Series and Transform

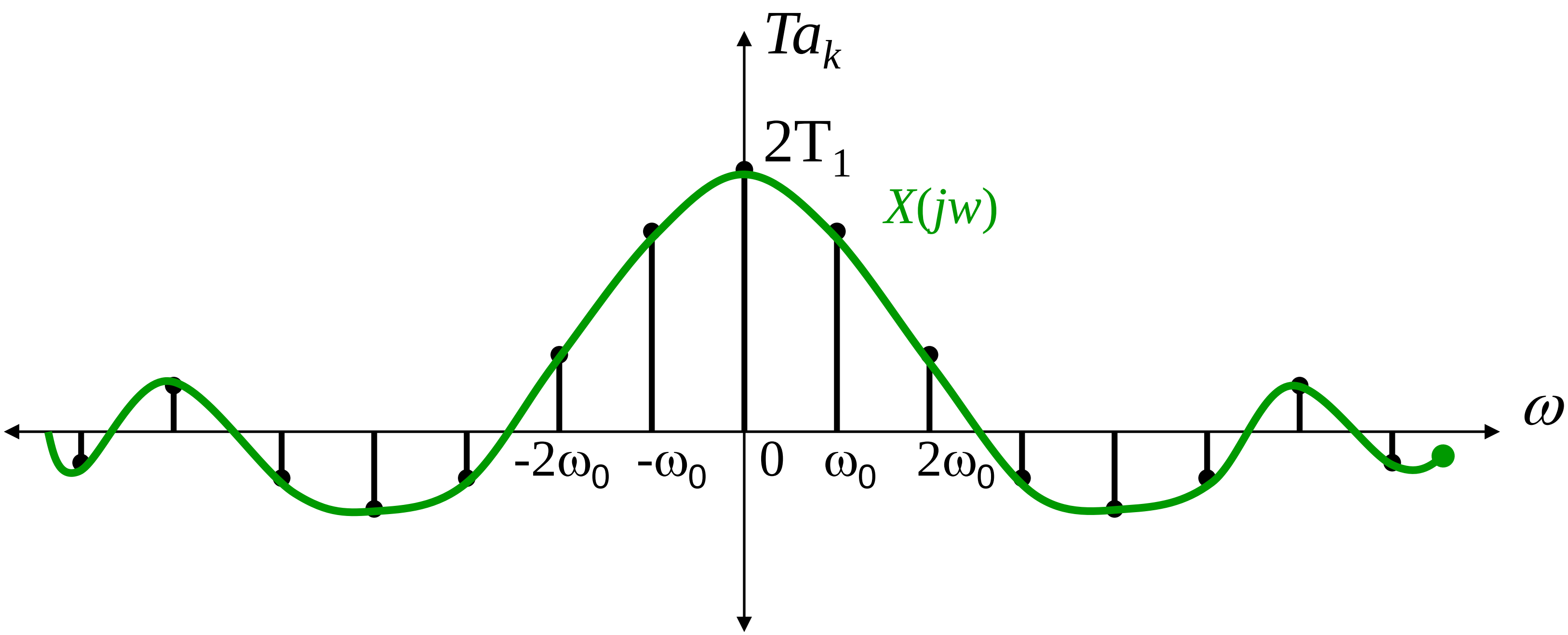
- The same sketch a_k on the ω-axis:



- On the ω-axis, the distance between two consecutive a_k's is $\omega_0=2\pi/T$, which is the fundamental frequency.

Bridge Between Fourier Series and Transform

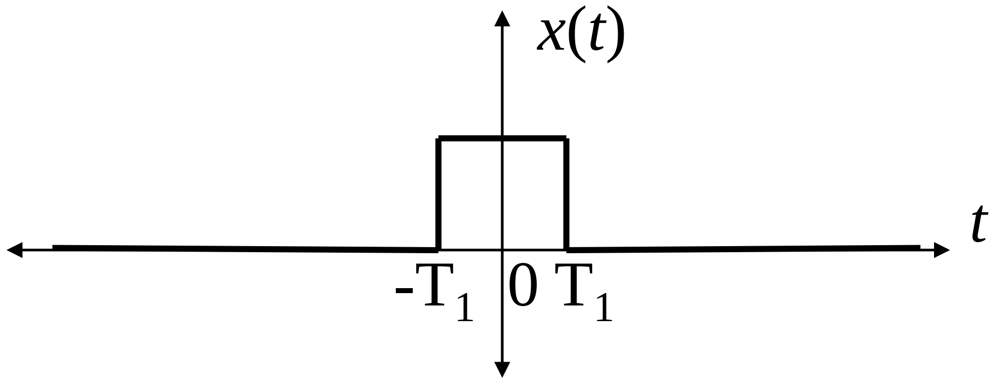
- The same sketch Ta_k on the ω-axis:



- The distance between two adjacent a_k's is $\omega_0 = 2\pi/T$.
- As T→◎, $\omega_0 \rightarrow 0$.
  - The distance between two consecutive a_k's becomes zero
  - The sketch of a_k becomes continuous
  - The continuous curve X(jw) is called as Fourier Transform

Bridge Between Fourier Series and Transform

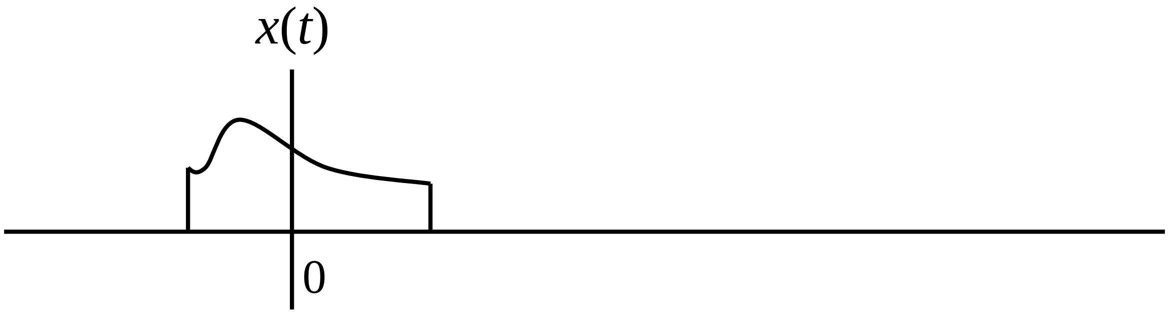
- On the other hand, as T→◎, the signal x(t) becomes an aperiodic signal

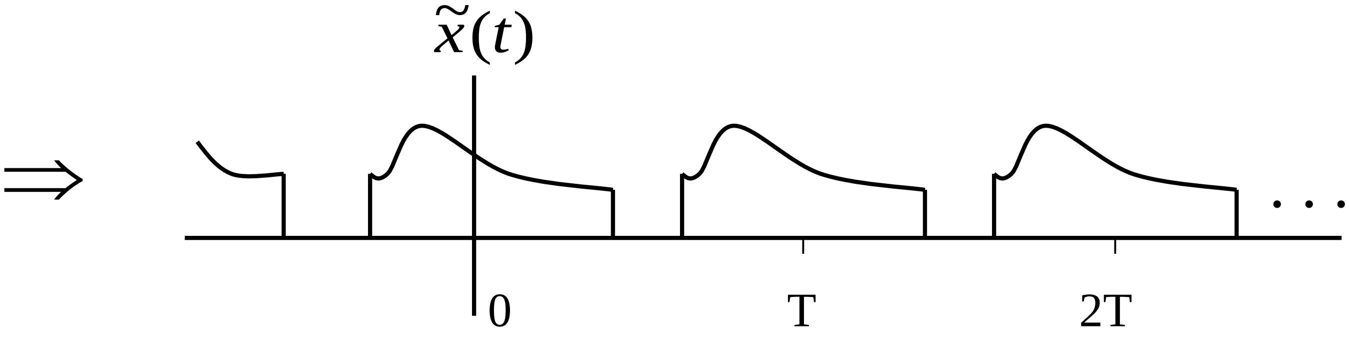


- Fourier Transform can represent an aperiodic signal in frequency domain

From CTFS to CTFT: Formal Derivation

- How can we use this formula for a non-periodic (aperiodic) function $x(t)$?





$$x(t) = \lim_{T \to \infty} \tilde{x}(t)$$

From CTFS to CTFT: Formal Derivation

- Given the relationships

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_T \tilde{x}(t) e^{-jk\omega_0 t} dt$$

$$x(t) = \lim_{T\to\infty} \tilde{x}(t)$$

derive the following CTFT formula

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

CTFT Formula – Fourier Transform Pair

- Forward Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- Inverse Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

  - $X(jw)$ represents the strength of frequency component at w in $x(t)$

Time Domain vs. Frequency Domain

- Fourier analysis (series or transform) is a tool to determine the frequency contents of a given signal
  - Conversion from time domain to frequency domain.

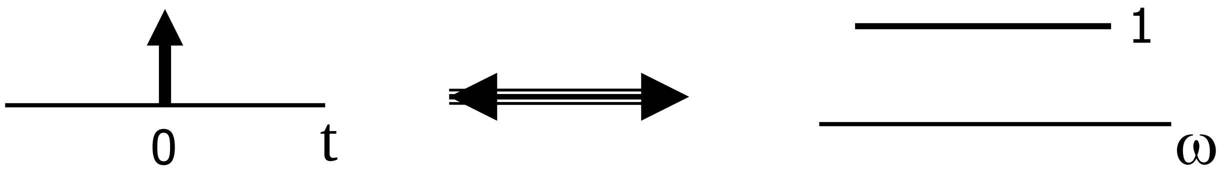
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- It is always possible to move back from frequency domain to time domain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Some Examples

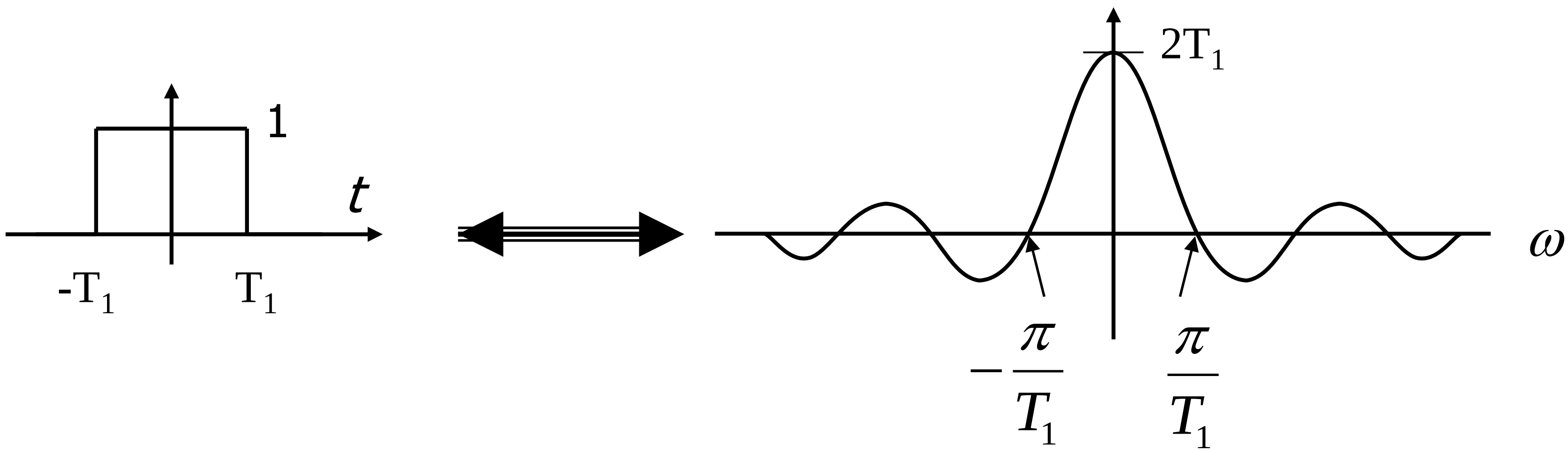
Ex 1) Impulse function → constant function



$$x(t) = \delta(t) \overset{F}{\longleftrightarrow} X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Some Examples

Ex 2) Rectangular pulse → sinc function



$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin \omega T_1}{\omega} = 2T_1 \ \mathrm{sinc}(\frac{\omega T_1}{\pi})$$

$$\text{where } \mathrm{sinc}(t) \triangleq \frac{\sin(\pi t)}{\pi t}$$

Note the inverse relationship between time and frequency domains

More Examples

Unified Framework for CTFS and CTFT:

Periodic Signals Can Also Be Represented as Fourier Transform

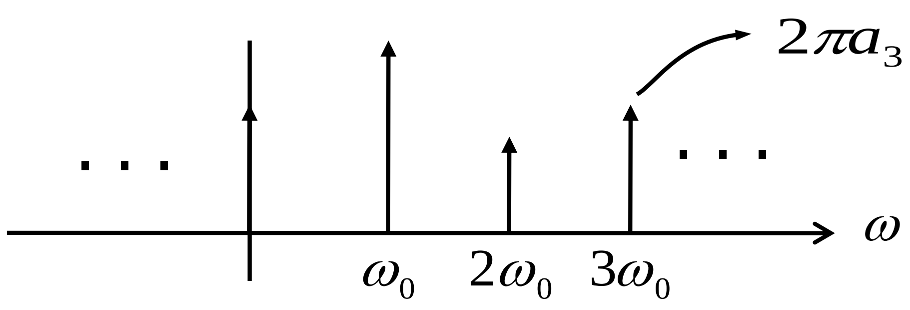
Fourier Transform for Periodic Signals

- Consider the inverse Fourier transform of

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

- So, we can deduce that

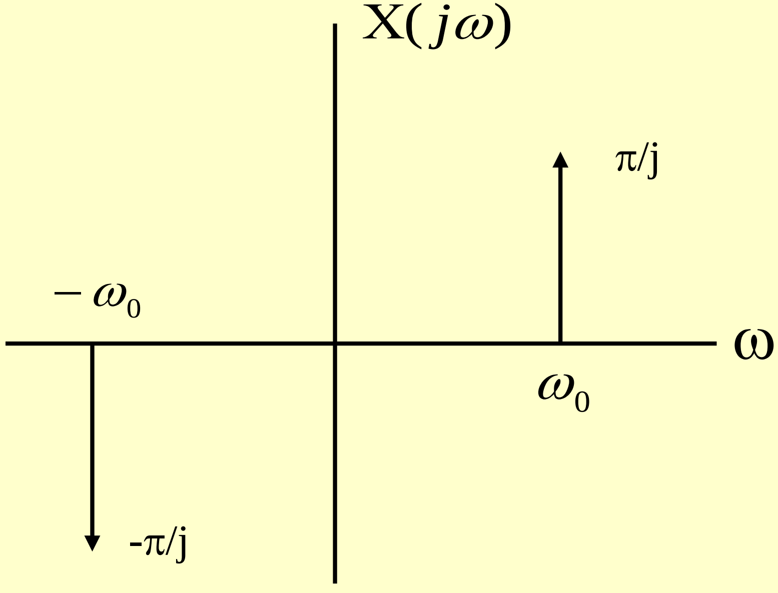
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \stackrel{\text{Fourier}}{\longleftrightarrow} X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$



Fourier Transform for Periodic Signals

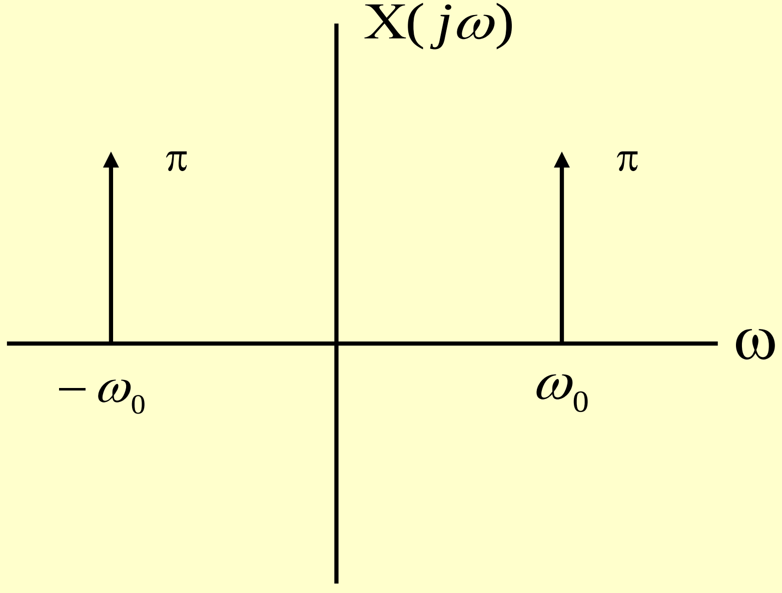
Ex 1) sin function

$$x(t) = \sin(\omega_0 t) \overset{F.S.}{\longleftrightarrow} a_1 = \frac{1}{2j},\ a_{-1} = \frac{1}{-2j}$$



Ex 2) cos function

$$x(t) = \cos(\omega_0 t) \overset{F.S.}{\longleftrightarrow} a_1 = a_{-1} = \frac{1}{2}$$

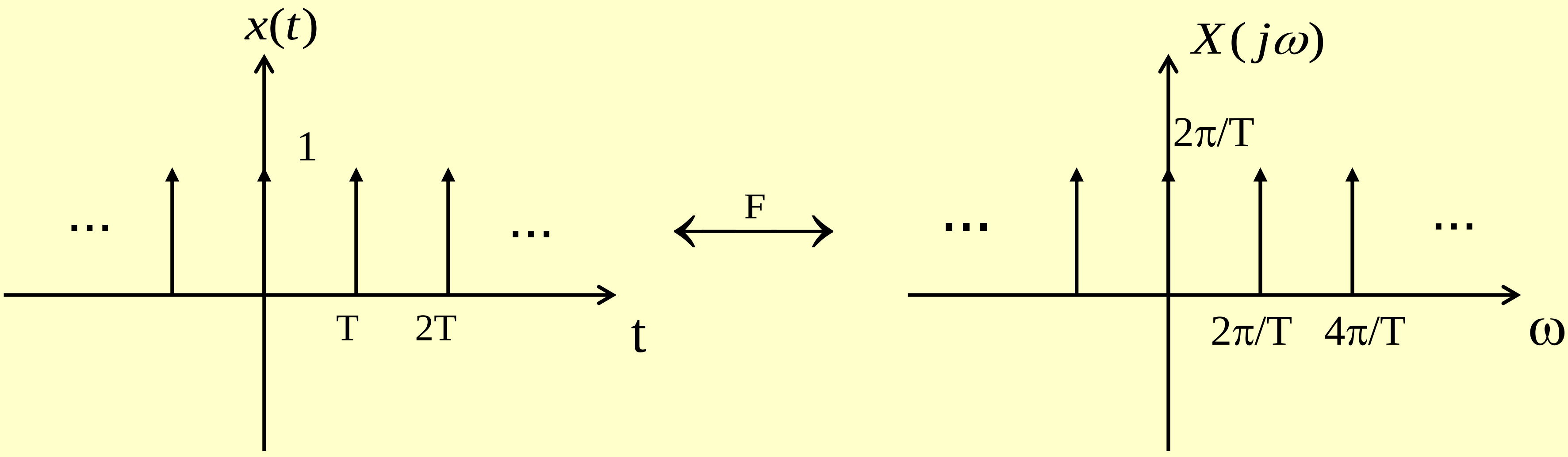


Fourier Transform for Periodic Signals

Ex 3) Fourier transform of impulse trains

$$x(t) = \sum_{-\infty}^{\infty} \delta(t - kT) \overset{F.S.}{\longleftrightarrow} a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) \cdot e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$\therefore X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{T})$$



Properties of CTFT

Properties of CTFT

1. Linearity

$$a \cdot x(t) + b \cdot y(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega)$$

2. Time shifting

$$x(t - t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega)$$

3. Conjugation and conjugate symmetry

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega)$$

$$X(j\omega) = X^*(-j\omega) \qquad [x(t) \text{ real}]$$

4. Differentiation and integration

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega \cdot \mathrm{X}(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Properties of CTFT

5. Time and frequency scaling

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X(\frac{j\omega}{a})$$

$$x(-t) \overset{F}{\longleftrightarrow} X(-j\omega)$$

6. Parseval's relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

7. Duality

$$g(t) \overset{\mathrm{F}}{\longleftrightarrow} G(j\omega) \Rightarrow G(jt) \overset{\mathrm{F}}{\longleftrightarrow} 2\pi g(-\omega)$$

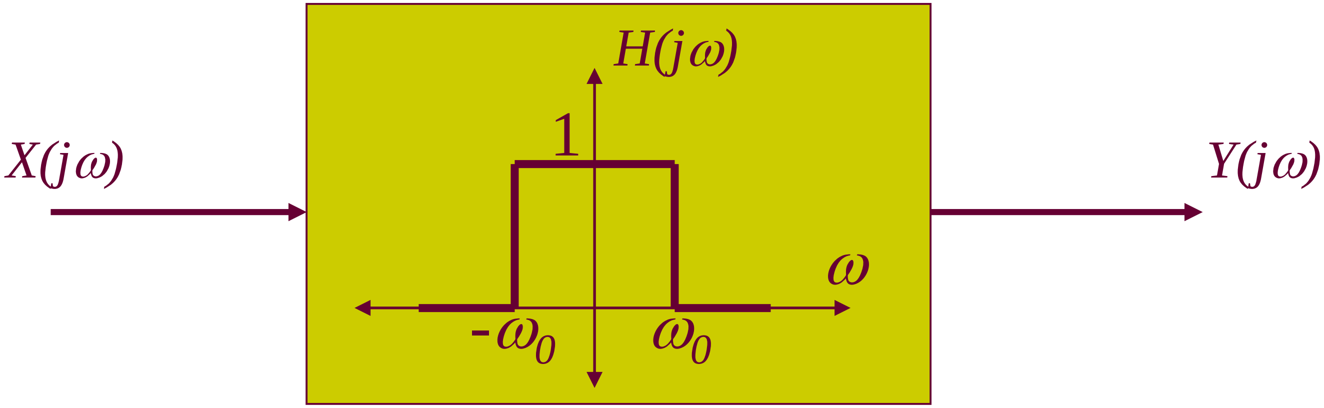
Convolution Property of CTFT

$$y(t) = h(t) * x(t) \overset{F}{\longleftrightarrow} Y(j\omega) = H(j\omega) \cdot X(j\omega)$$

- Two approaches for proof and understanding
  1. LTI interpretation
     - Note that the frequency response H(jw) is just the CTFT of the impulse response h(t).
  2. Direct equation manipulation

Convolution Property of CTFT

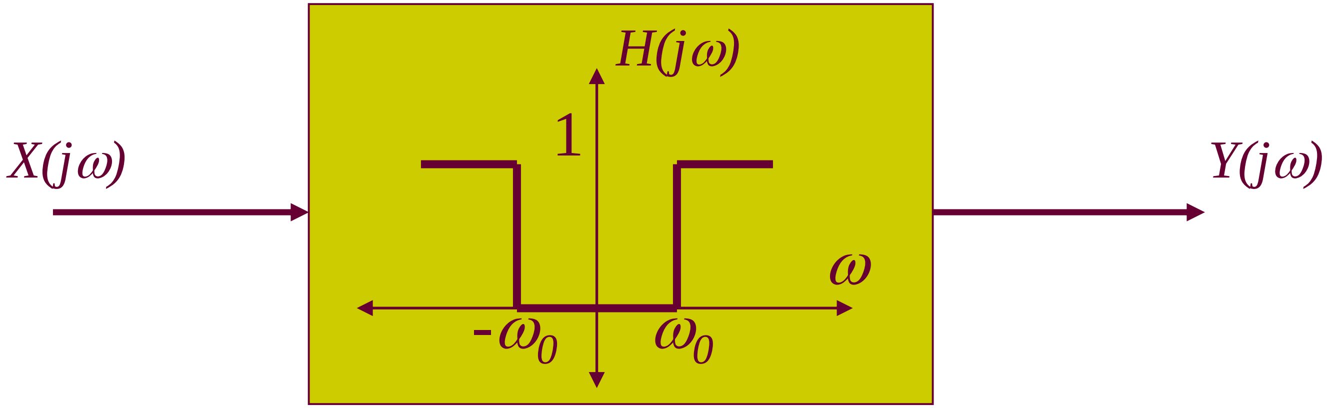
- Lowpass Filter



$$Y(j\omega) = \begin{cases} 0, & \omega < -\omega_0 \\ X(j\omega), & -\omega_0 < \omega < \omega_0 \\ 0, & \omega_0 < \omega \end{cases}$$

Convolution Property of CTFT

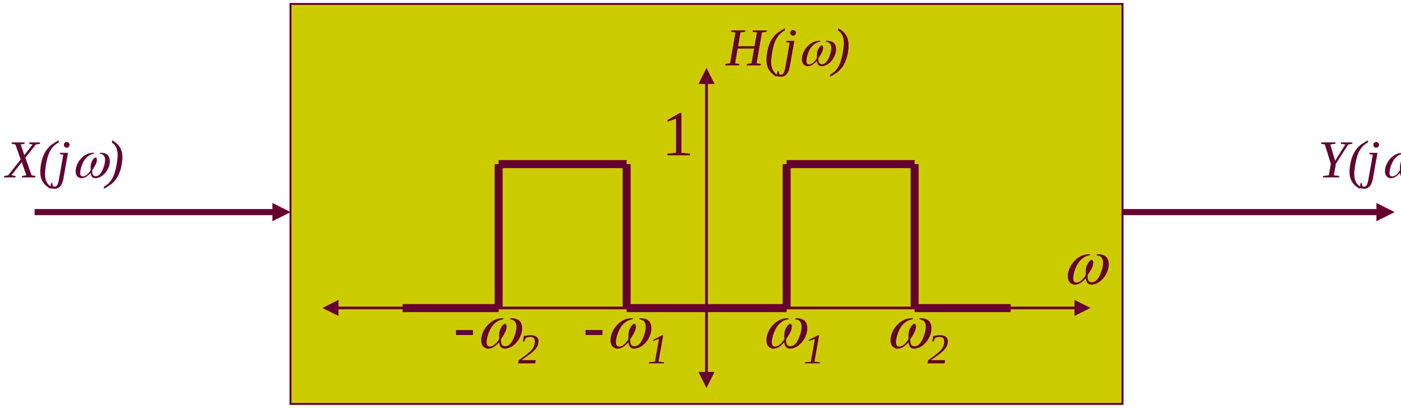
- Highpass Filter:



$$Y(j\omega)=\begin{cases} X(j\omega), & \omega<-\omega_0 \\ 0, & -\omega_0<\omega<\omega_0 \\ X(j\omega), & \omega_0<\omega \end{cases}$$

Convolution Property of CTFT

- Bandpass Filter:



$$Y(j\omega) = \begin{cases} X(j\omega), & -\omega_1 < \omega < -\omega_2 \\ X(j\omega), & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

Examples

CTFT Table

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)

CTFT Table

Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \le \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\,\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\,\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t)\begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—

CTFT Table

$\delta(t)$	1	—
$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{a + j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$	—
$\dfrac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^n}$	—

Multiplication Property of CTFT

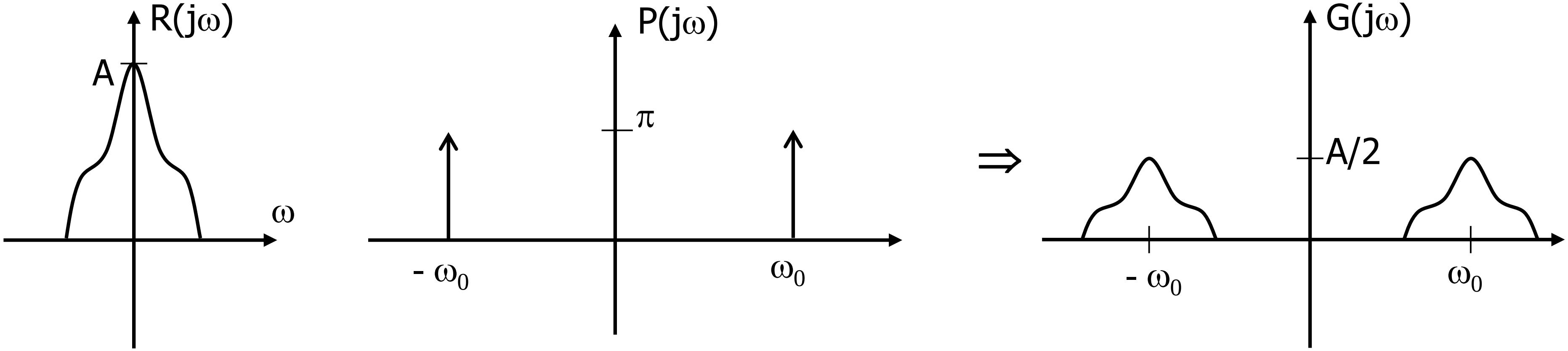
$$r(t) = s(t) \cdot p(t) \xleftrightarrow{F} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega-\theta))d\theta$$

- This is a dual of the convolution property

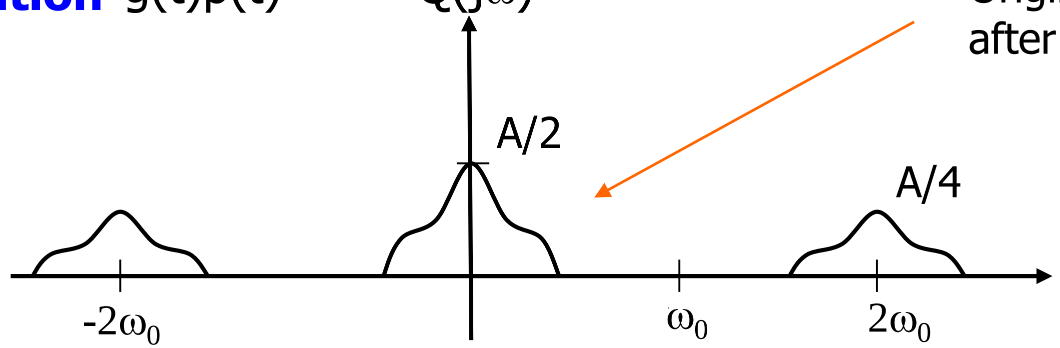
Multiplication Property of CTFT

Idea of AM (amplitude modulation)

FT of r(t) FT of $p(t)=\cos\omega_0 t$ **modulation** $g(t) = r(t)\cdot p(t)$



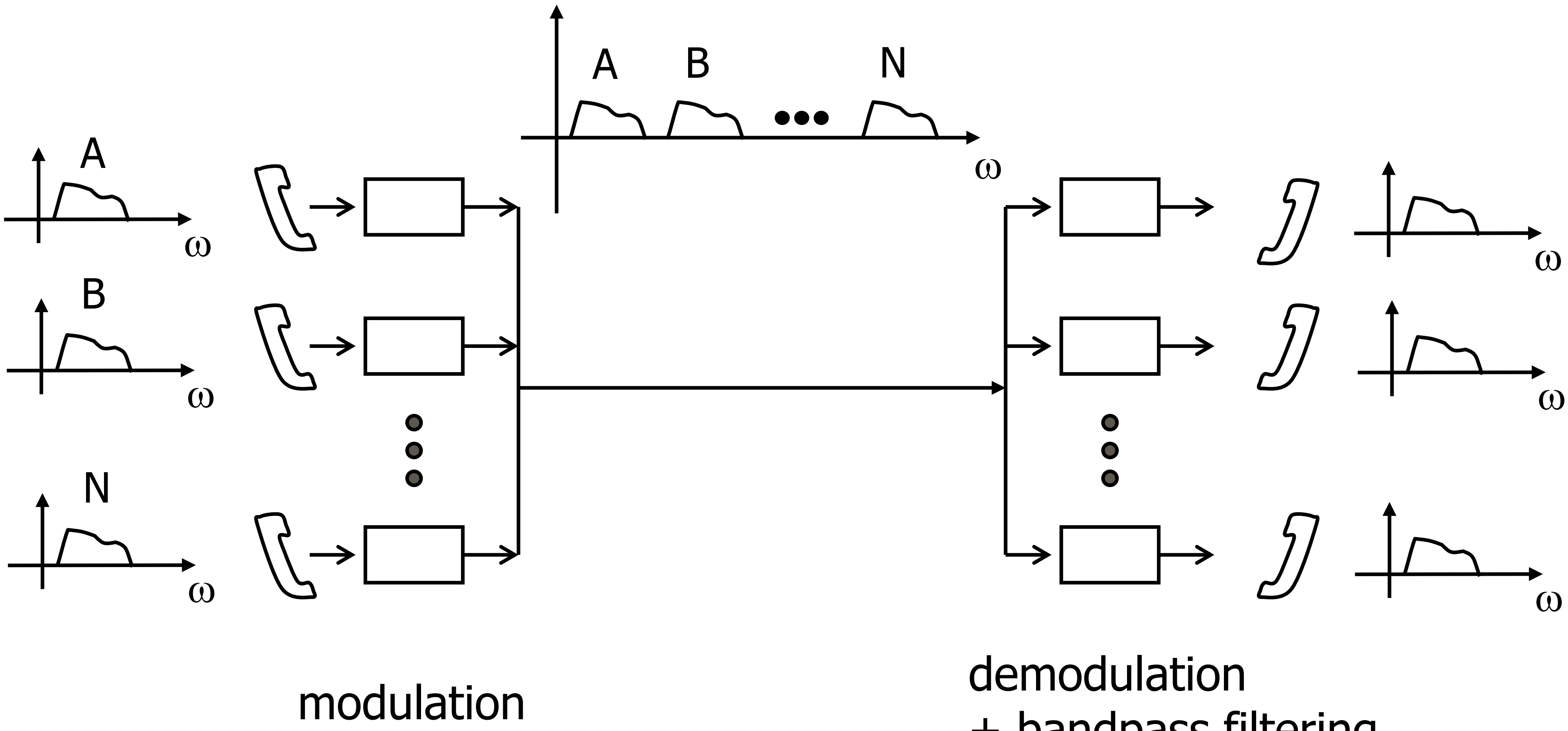
demodulation $g(t)p(t) \overset{F}{\longleftrightarrow} Q(j\omega)$



Original signal is recovered after a low-pass filter

Multiplication Property of CTFT

A communication system



Causal LTI Systems Described by Differential Equations

Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- The DE describes the relation between the input x(t) and the output y(t) implicitly
- In this course, we are interested in DEs that describe causal LTI systems
- Therefore, we assume the initial rest condition

$$\text{If } x(t) = 0 \text{ for } t < t_0, \text{ then } y(t) = 0 \text{ for } t < t_0$$

which also implies

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1} y(t_0)}{dt^{N-1}} = 0$$

Frequency Response

- What is the frequency response H(jw) of the following system?

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- It is given by

$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$

Example

Q) $$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t),$$

$$x(t)=e^{-t}u(t).$$

A) $$Y(j\omega)=H(j\omega)X(j\omega)$$

$$=\left[\frac{j\omega+2}{(j\omega)^2+4(j\omega)+3}\right]\left[\frac{1}{j\omega+1}\right]$$

$$=\frac{j\omega+2}{(j\omega+1)^2(j\omega+3)}$$

$$=\frac{\frac{1}{4}}{j\omega+1}+\frac{\frac{1}{2}}{(j\omega+1)^2}-\frac{\frac{1}{4}}{j\omega+3}$$

$$\Rightarrow y(t)=\left[\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}-\frac{1}{4}e^{-3t}\right]u(t)$$