Discrete-Time Fourier Transform

Chang-Su Kim

	continuous time	discrete time
periodic (series)	CTFS	DTFS
aperiodic (transform)	CTFT	**DTFT**

DTFT Formula and Its Derivation

DTFT Formula

- DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- cf) CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Note that in DT case, $X(e^{j\omega})$ is periodic with period 2π and the inverse transform is defined as a integral over one period

Examples

- Find the Fourier transforms of

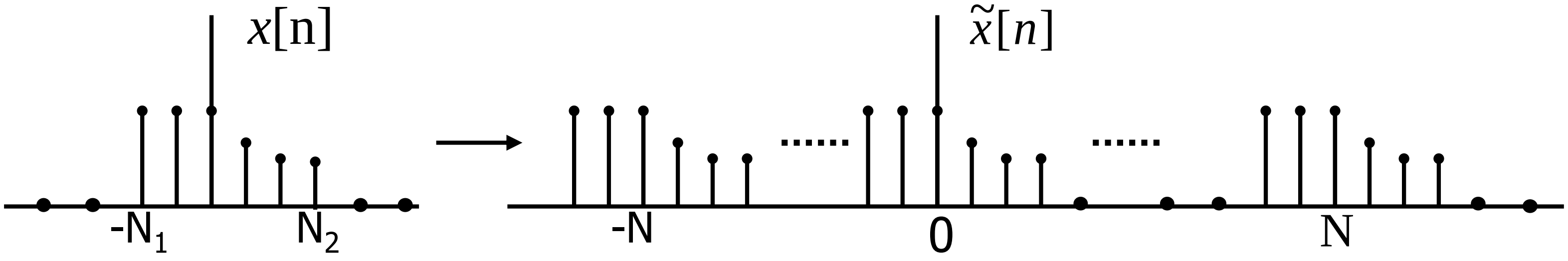
(a) $\left(\frac{1}{2}\right)^{n-1} u[n-1]$

(b) $\delta[n-1]+\delta[n+1]$

- Find the inverse Fourier transform of

(a) $X(e^{jw}) = \begin{cases} 2j, & 0 < w \le \pi \\ -2j, & -\pi < w \le 0 \end{cases}$

Derivation of DTFT from DTFS



$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(\frac{2\pi}{N})n}$$

As $N \to \infty,\ \tilde{x}[n] \to x[n]$

and the DTFS formula becomes the desired DTFT formula

DTFT of Periodic Functions

- Periodic functions can also be represented as Fourier Transforms

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \xleftrightarrow{F} X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Examples

- DTFT of periodic functions

(a) cosine function

$$x[n] = \cos w_0 n$$

(b) periodic impulse train

$$y[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

Selected Properties of DTFT

Shift in Frequency

$$e^{jw_0 n} x[n] \overset{F}{\longleftrightarrow} X(e^{j(w-w_0)})$$

$$(-1)^n x[n] \overset{F}{\longleftrightarrow} X(e^{j(w-\pi)})$$

- This property can be used to convert a lowpass filter to a highpass one, or vice versa

Differentiation in Frequency

$$nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{jw})}{dw}$$

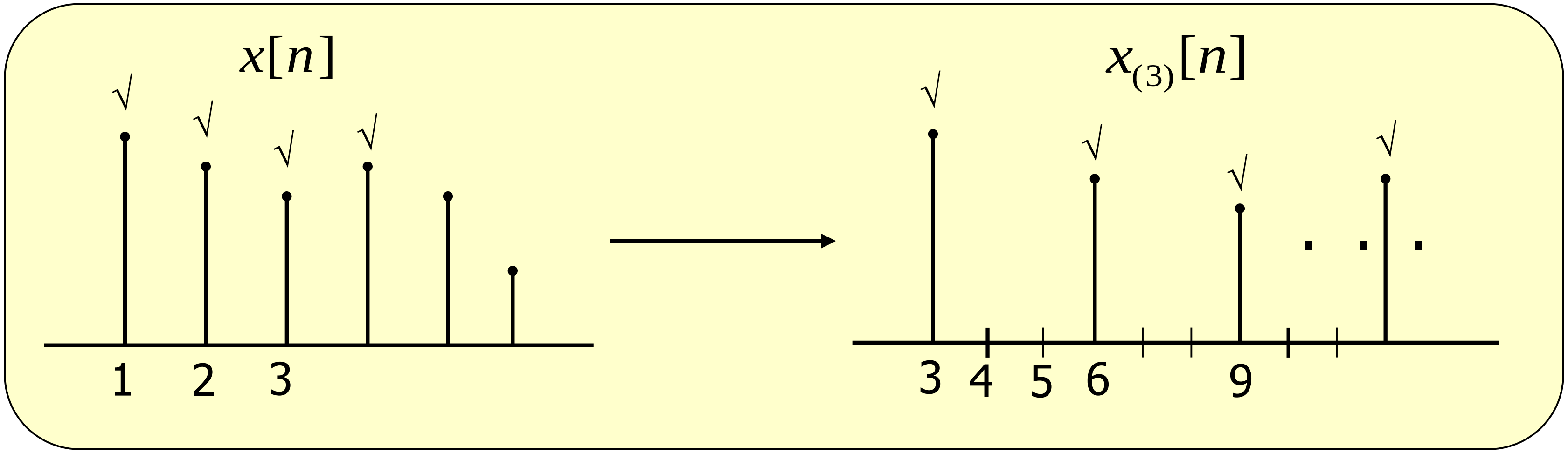
Parseval's Relation

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \int_{2\pi} \left| X\left(e^{jw}\right) \right|^2 dw$$

Time Expansion

- For a natural number k, we define

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is an integer multiple of } k \\ 0 \quad , & \text{otherwise} \end{cases}$$



$$\Rightarrow \quad x_{(k)}[n] \stackrel{F}{\longleftrightarrow} X(e^{jk\omega})$$

Convolution

$$y[n] = x[n] * h[n] \xleftrightarrow{F} Y(e^{jw}) = X(e^{jw})H(e^{jw})$$

Multiplication

$$y[n] = x_1[n]x_2[n] \xleftrightarrow{F} Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

- Multiplication in time domain corresponds to the **periodic convolution** in frequency domain

Summary of Fourier Series and Transform Expressions

All the Four Formulas

	CT	DT
Periodic (series)	CTFS $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$ $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$	DTFS $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$
Aperiodic (transform)	CTFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	DTFT $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Properties

	Time		Frequency	
$x[n],\ x(t)$	aperiodic	$\Leftrightarrow$	continuous	$X(e^{j\omega}),\ X(j\omega)$
$x[n],\ x(t)$	periodic	$\Leftrightarrow$	discrete	$X(e^{j\omega}),\ X(j\omega)$
$x[n]$	discrete	$\Leftrightarrow$	periodic	$X(e^{j\omega})$
$x(t)$	continuous	$\Leftrightarrow$	aperiodic	$X(j\omega)$

	CT	**DT**
Periodic (series)	CTFS $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ $\Rightarrow X(j\omega)=2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$	DTFS $x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k=\frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(\frac{2\pi}{N})n}$ sampling of continuous functions => discrete $\Rightarrow X(e^{j\omega})=2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega-k\omega_0)$
Aperiodic (transform)	CTFT $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	DTFT $x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ $X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Dualities

	CT	DT
Periodic (series)	CTFS $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$ $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$	DTFS $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$
Aperiodic (transform)	CTFT $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$	DTFT $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Dualities

	CT	DT
Periodic (series)	CTFS $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$ $$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$	DTFS $$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(\frac{2\pi}{N})n}$$ ⇕ $$a_k = \frac{1}{N}\sum_{n=\langle N \rangle} x[n] e^{-jk(\frac{2\pi}{N})n}$$
Aperiodic (transform)	CTFT $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$ $$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$	DTFT $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$ $$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Dualities

	CT	DT
Periodic (series)	CTFS $x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$ $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$	DTFS $x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(\frac{2\pi}{N})n}$
Aperiodic (transform)	CTFT $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$ $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$	DTFT $x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ $X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Causal LTI Systems Described by Difference Equations

Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- The DE describes the relation between the input x[n] and the output y[n] implicitly
- In this course, we are interested in DEs that describe causal LTI systems
- Therefore, we assume the initial rest condition

$$\text{If } \; x[n]=0 \text{ for } \; n<n_0, \text{ then } \; y[n]=0 \text{ for } \; n<n_0$$

Frequency Response

- What is the frequency response $H(e^{jw})$ of the following system?

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- It is given by

$$H(e^{jw}) = \frac{\sum_{k=0}^{M} b_k e^{-jkw}}{\sum_{k=0}^{N} a_k e^{-jkw}}$$

Example

Q) $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n],$

$x[n]=\left(\frac{1}{4}\right)^n u[n].$ What is $y[n]$?

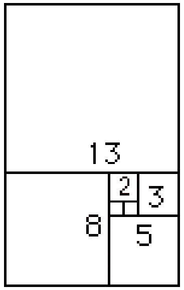
Analogy between Differential Equations and Difference Equations

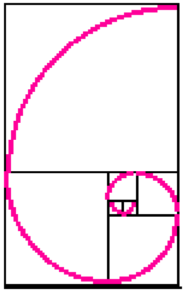
- Many properties, learned in differential equations, can be applied to solve interesting problems described by difference equations

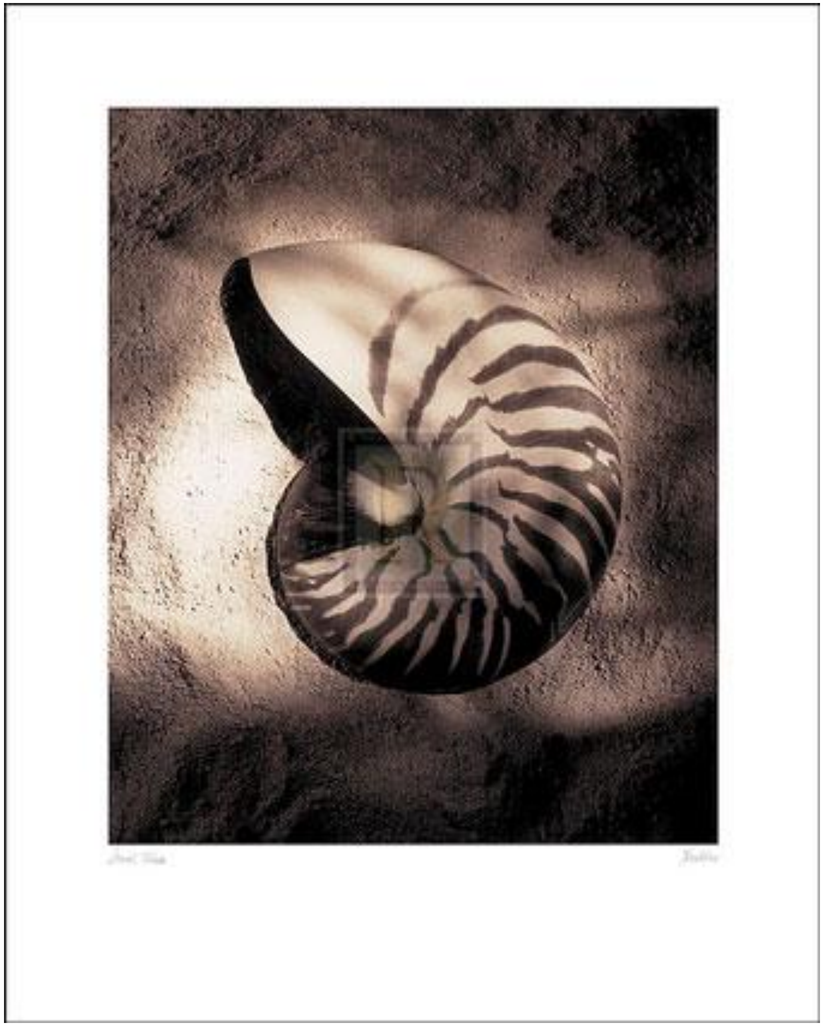
- Fibonacci sequence
  - $a_0 = a_1 = 1$
  - $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$
  - Golden ratio = 1.61803

Golden Ratio

- Golden rectangle is said to be the most visually pleasing geometric form







Golden Ratio

- Golden rectangle is said to be the most visually pleasing geometric form