The z-Transform

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Some figures have been excerpted from

- The lecture notes of Dr. Weiss in MIT (http://umech.mit.edu/weiss/lectures.html)

z-Transform is an extension of DTFT

- z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z-Transform vs. DTFT

$$X(z)|_{z=e^{jw}} = \text{DTFT of } x[n]$$

Why do we need the extension?

- Consider the DTFT pair

$$a^n u[n] \xleftrightarrow{F} \frac{1}{1-ae^{-jw}} \qquad |a|<1$$

  - What happens if |a| ≥ 1?

- z-Transform pair

$$a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}, \qquad |z|>|a|$$

ROC

(region of convergence)

- z-Transform can be applied to a broader class of signals than DTFT
  - It is useful in studying a broader class of systems
  - It is used to analyze the causality and stability of a system

Inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

- Integral along a circular path in the complex number plane
  - Its proper evaluation requires some knowledge on complex integral
  - For example, refer to
    - R. V. Churchill and J. W. Brown, Complex Variables and Applications, 5^{th} edition, McGraw-Hill
- We do not use this formula. Instead, we decompose X(z) into a number of terms, each of which can be inverse transformed using Table 10.2.

Parts of Table 10.2

z-Transform Pair			ROC
(1) $\delta[n-m]$	$\Leftrightarrow$	z^{-m}	All z except z= 0 or ∞
(2) $\alpha^{n}u[n]$	$\Leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
(3) $-\alpha^{n}u[-n-1]$	$\Leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
(4) $[\cos \omega_0 n]u[n]$	$\Leftrightarrow$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| >1$
(5) $[\sin \omega_0 n]u[n]$	$\Leftrightarrow$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\|z\| > 1$

ROC should be specified

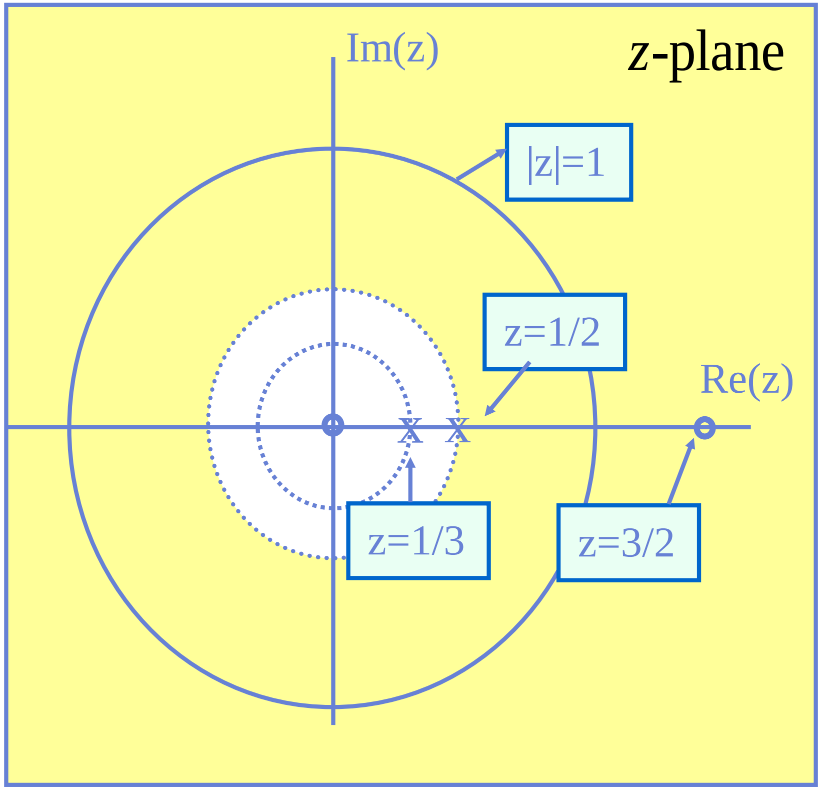
$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

ROC should be specified

Ex) $x[n] = 7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]$

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$

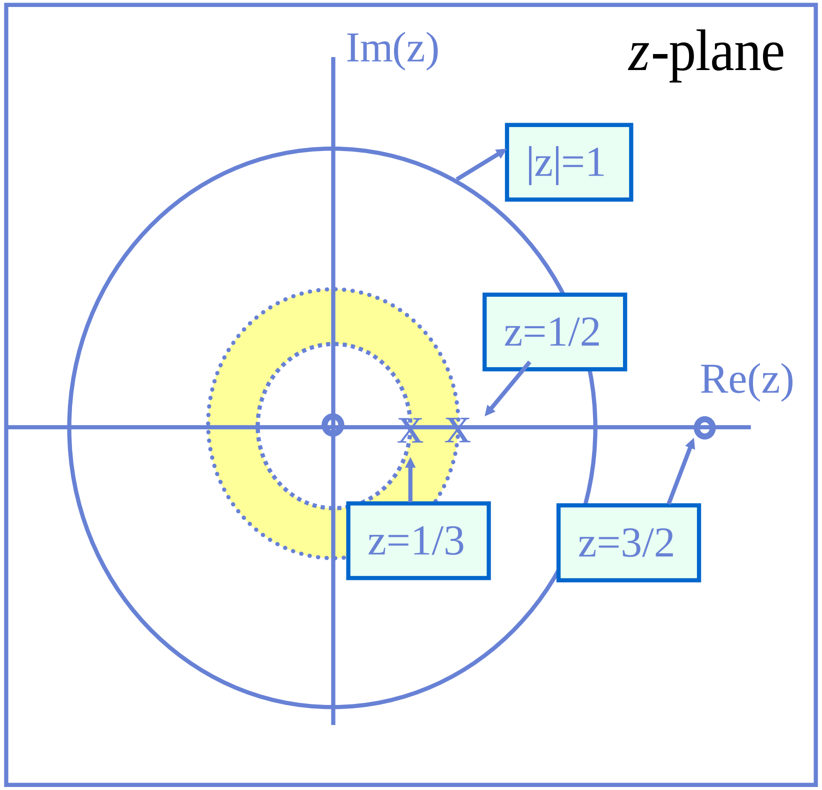


There are other sequences, which generate the same X(z) but with different ROC's

ROC should be specified

Ex) $x[n] = ?$

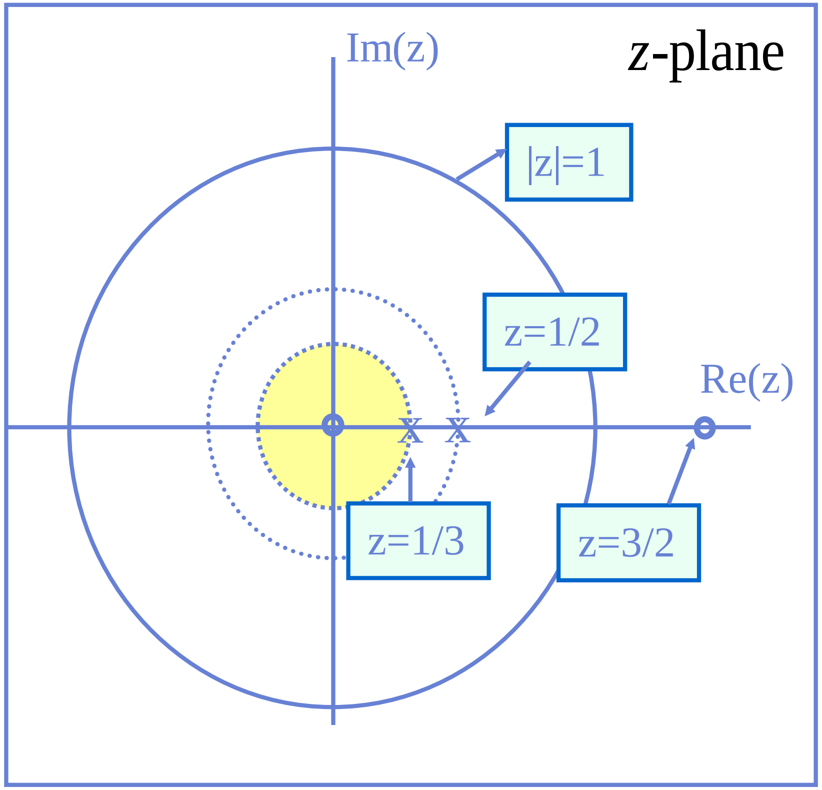
$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$



ROC should be specified

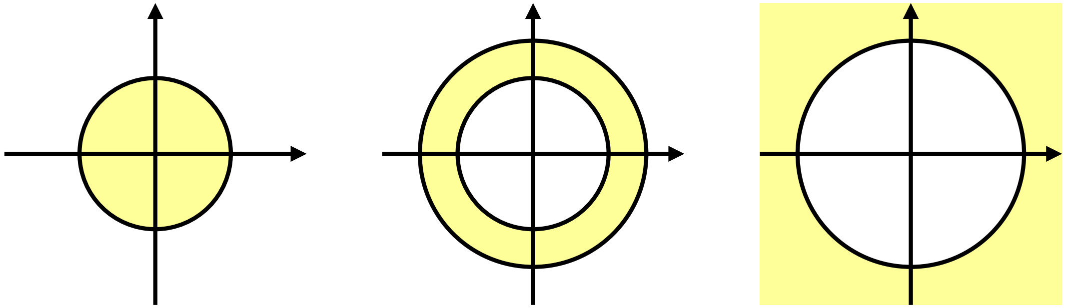
Ex) $x[n] = ?$

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| < \frac{1}{3}$$



Properties on ROC

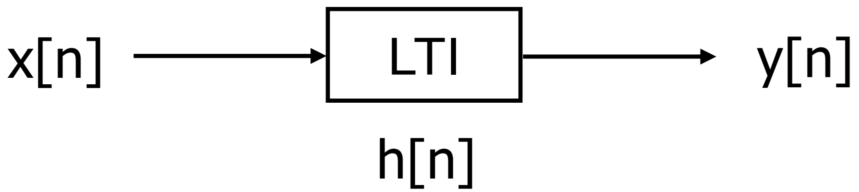
- ROC of X(z) consists of a single ring in the z-plane centered at the origin.



- ROC does not contain any poles.
- Suppose that X(z) is rational.
  1. Its ROC is bounded by poles.
  2. If x[n] is right sided, ROC is the region in the z-plane outside the outermost pole.
  3. If x[n] is left sided, ROC is the region inside the innermost nonzero pole.

Selected Properties of Z-Transform

- z^n is the eigenfunction of LTI systems



If $x[n] = z^n$, $y[n] = H(z)z^n$.

$H(z)$ is called system function or transfer function.

Selected Properties of Z-Transform

- z^{-1} is a delay system

$$x[n-n_0] \longleftrightarrow z^{-n_0} X(z)$$

Selected Properties of Z-Transform

- Time reversal

$$x[n] \longleftrightarrow X(z), \text{ with ROC} = \text{R}$$

$$x[-n] \longleftrightarrow X\left(\frac{1}{z}\right), \text{ with ROC} = \frac{1}{\text{R}}$$

Selected Properties of Z-Transform

- Convolution

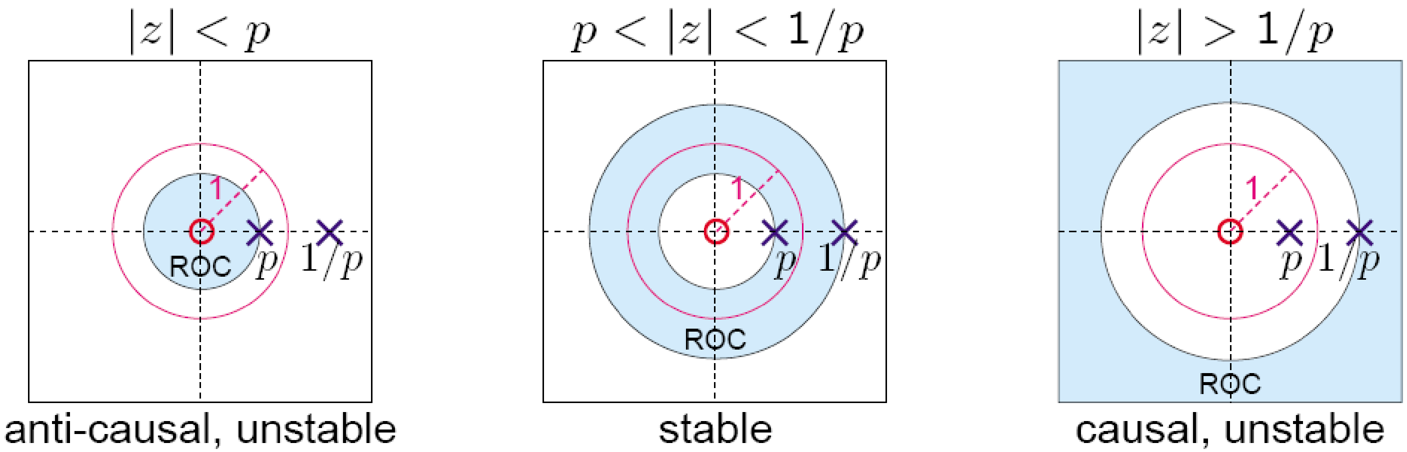
$$x_1[n] \longleftrightarrow X_1(z), \text{ with ROC} = \mathrm{R}_1$$

$$x_2[n] \longleftrightarrow X_2(z), \text{ with ROC} = \mathrm{R}_2$$

$$x_1[n] * x_2[n] \longleftrightarrow X_1(z) X_2(z), \text{ with ROC containing } \mathrm{R}_1 \cap \mathrm{R}_2$$

Characterization of LTI Systems Using z-Transform

- Causality
  - if the ROC of the transfer function is the exterior of a circle, including infinity
- Stability
  - if the ROC of the transfer function contains the unit circle |z|=1.



LTI Systems Specified by Linear Constant-Coefficient Difference Equations

Ex) $y[n]+\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n]-\frac{7}{4}x[n-1]-\frac{1}{2}x[n-2]$