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Normalized cuts and image segmentation

- We cut a graph $G=(V,E)$ by removing edges

$$V = A \cup B \quad \text{and} \quad A \cap B = \phi.$$

- $\mathrm{Cut}(A,B) \triangleq \sum_{u\in A, v\in B} w(u,v)$

- $\mathrm{assoc}(A,V) \triangleq \sum_{u\in A, v\in V} w(u,v)$

- Criterion

$$\mathrm{Ncut}(A,B) = \frac{\overbrace{\mathrm{cut}(A,B)}^{\text{should be small}}}{\underbrace{\mathrm{assoc}(A,V)}_{\text{b.g}}} + \frac{\overbrace{\mathrm{cut}(A,B)}^{\text{small}}}{\underbrace{\mathrm{assoc}(B,V)}_{\text{b.g}}}$$

necessary to avoid small-cut problem

$\cdots$ ①

- Definition of variable x

$$x_i = \begin{cases} 1 & i \in A \\ -1 & i \in B \end{cases}$$

$$b = \frac{\sum_{x_i>0} d_i}{\sum_{x_i<0} d_i}$$

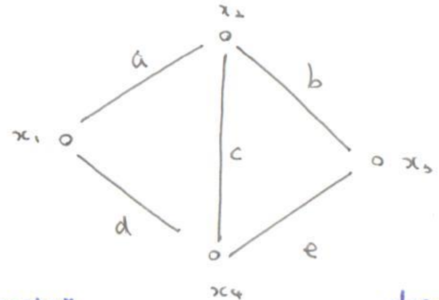
- $y_i \triangleq (1+x_i) - b(1-x_i)$

x		y
1	$\Leftrightarrow$	2
-1		$-2b$

Then $\boxed{y^T D \mathbf{1} = 0}$ constraint on y

We can rewrite Ncut in ① as

$$\mathrm{Ncut}(y) = \frac{y^T(D-W)y}{y^T D y}$$



incidence matrix

$$W = \begin{bmatrix} 0 & a & 0 & d \\ a & 0 & b & c \\ 0 & b & 0 & e \\ d & c & e & 0 \end{bmatrix}$$

degree matrix

$$D = \begin{bmatrix} a+d & & & \\ & a+b+c & & \\ & & b+e & \\ & & & d+c+e \end{bmatrix}$$

Problem

$$\text{minimize} \quad \frac{y^T(D-W)y}{y^T D y} \quad \text{subject to} \quad y^T D \mathbf{1} = 0 \quad \cdots (2)$$

$$\text{and} \quad \boxed{y_i = 1 \text{ or } -b}$$

ignore approximately

$$\frac{y^T(D-W)y}{y^T D y} = \frac{z^T D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}} z}{z^T z} \qquad z = D^{\frac{1}{2}} y \text{ or } y = D^{-\frac{1}{2}} z$$

$$D^{-\frac{1}{2}}(D-W)D^{-\frac{1}{2}} = Q\Lambda Q^T$$

Laplacian: positive semidefinite

$$(D-W)\mathbf{1} = 0$$

$$\Lambda = \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad Q = \begin{bmatrix} z_1 & \cdots & z_n \end{bmatrix}$$

$$z_1 = D^{\frac{1}{2}} \mathbf{1}$$

$$= \frac{z^T Q \Lambda Q^T z}{z^T z}$$

$$= \frac{\alpha^T \Lambda \alpha}{\alpha^T \alpha} \qquad \alpha = Q^T z \text{ or } z = Q\alpha$$

$$(2) \approx \quad z^T D^{\frac{1}{2}} \mathbf{1} = z^T z_1 = 0$$

$$\text{or} \quad \alpha^T Q^T Q \alpha_1 = \alpha^T \alpha_1 = 0$$

$$\text{where} \quad \alpha_1 = Q^T z_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\therefore \quad \min \frac{\alpha^T \Lambda \alpha}{\alpha^T \alpha}$$

$$= \min \frac{\lambda_1 \beta_1^2 + \cdots + \lambda_n \beta_n^2}{\beta_1^2 + \cdots + \beta_n^2} \qquad (\beta_1 = 0)$$

$$= \min \frac{\lambda_2 \beta_2^2 + \cdots + \lambda_n \beta_n^2}{\beta_2^2 + \cdots + \beta_n^2}$$

$$= \lambda_2 \quad \text{when} \quad \alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\text{or} \quad z = Q \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = z_2$$

$$\text{or} \quad y = D^{-\frac{1}{2}} z_2$$

In other words, it is minimized when y is the generalized eigenvector of $\frac{y^T(D-W)y}{y^T D y}$ corresponding to the 2nd smallest eigenvalue.