Chap 2. Vector Spaces

A vector space V

① Vector addition $x, y \in V$ $x+y$ is defined.

② Scalar multiplication $x \in V$, $c \in \mathbb{R}$ cx is defined

(can be any field such as $\mathbb{C}$)

Ex 1) $\mathbb{R}^2$

- $(x_1, x_2) + (y_1, y_2) = (x_1+y_1, x_2+y_2)$
- $c(x_1, x_2) = (cx_1, cx_2)$

Formal definition

1. $x + y = y + x$
2. $x + (y + z) = (x+y)+z$
3. There is a unique zero vector 0 s.t $x + 0 = x$ for all x.
4. For each x, there is a unique vector $-x$ s.t. $x + (-x) = 0$.
5. $1x = x$
6. $(c_1 c_2) x = c_1(c_2 x)$
7. $c(x+y) = cx + cy$
8. $(c_1 + c_2) x = c_1 x + c_2 x$.

Ex 2) $\mathbb{R}^\infty$

$x = (x_1, x_2, x_3, \cdots)$

$y = (y_1, y_2, y_3, \cdots)$

$\Rightarrow$ $x+y = (x_1+y_1, x_2+y_2, x_3+y_3, \cdots)$

$cx = (cx_1, cx_2, cx_3, \cdots)$

Ex 3) A space of polynomials of order less than n

$x = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n$

$y = d_0 + d_1 t + d_2 t^2 + \cdots + d_n t^n$

$x + y = c_0 + d_0 + (c_1+d_1)t + \cdots + (c_n+d_n)t^n$

$\alpha x = \alpha c_0 + \alpha c_1 t + \cdots + \alpha c_n t^n$.

Ex 4) A space of functions defined on $[0, 1]$

$f(t) = \sin t$ $\quad g(t) = e^t$

$\Rightarrow$ $f(t) + g(t) = e^t + \sin t$

$2f(t) = 2\sin t$

Subspace

A subspace S of a vector space is a nonempty subset that satisfies

(i) $x, y \in S \Rightarrow x + y \in S$

(ii) for all scalar c and $x \in S$, $cx \in S$

Notice. A subspace is a space, satisfying the eight requirements.

e.g. $\bar{0} \in S$ because $(1+0)x = 1x + 0x = 1x$

$\therefore \bar{0} = 0x \in S$

Ex) $A = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$

A is a subset of $\mathbb{R}^2$, but not a subspace.

$[\because -1(1, 1) = (-1, -1) \notin A.]$

$B = \{(x, y) \in \mathbb{R}^2 : (x, y)$ is in the first or the third quadrant$\}$

$(1, 2) + (-2, -1) = (-1, 1) \notin B$

$\therefore B$ is not a subspace.

Column Space

$$Ax = b$$

For example,

$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If it has a solution $(u, v)^T$,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} u \\ 5u + 4v \\ 2u + 4v \end{bmatrix} = \underbrace{u \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} + v \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}}_{\text{linear combination of the columns of } A.}$$

In general, the system $Ax = b$ has a solution iff b can be expressed as a linear combination of the columns of A.

Column space of A of size $m \times n$

$$R(A) = \{ b \in \mathbb{R}^m ; \ b \text{ is a linear combination of the columns of } A \}.$$

$R(A)$ is a subspace of $\mathbb{R}^m$

$\because$ $b_1, b_2 \in R(A)$

$\Rightarrow b_1 = Ax_1,\ b_2 = Ax_2 \Rightarrow b_1 + b_2 = A(x_1 + x_2)$

$\Rightarrow b_1 + b_2 \in R(A)$

$cb_1 = A(cx_1) \Rightarrow cb_1 \in R(A)$

Ex) • $A = \bar{O}$

$R(A) = \{(0, 0, \cdots, 0)^T\}$

• $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$R(A) = \mathbb{R}^4$

<u>Nullspace of A</u>

$N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$

(i) $x_1, x_2 \in N(A)$

$\Rightarrow Ax_1 = Ax_2 = 0$

$\Rightarrow A(x_1 + x_2) = 0$

$\Rightarrow x_1 + x_2 \in N(A)$

(ii) $Acx_1 = cAx_1 = 0$

$\Rightarrow cx_1 \in N(A)$.

$\therefore$ $N(A)$ is a subspace of $\mathbb{R}^n$.

Ex) $B = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}$, $N(B)$?

Solve $Bx = 0$. Thus we use the Gaussian elimination

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$x = (x_1, x_2, x_3)$, x_3 is a free variable.

$4x_2 + 4x_3 = 0 \to x_2 = -x_3$

$x_1 + x_3 = 0 \to x_1 = -x_3$

$\therefore$ $x = (-x_3, -x_3, x_3)^T$

$\therefore$ $N(B) = \{ x : x = c(1, 1, -1)^T, c \in R \}$.

a line in $\mathbb{R}^3$.

2.2 m Equations in n unknowns

$$\underset{(m\times n)}{A}\,\underset{(n\times 1)}{x} = \underset{(m\times 1)}{b}$$

A) Homogeneous System $Ax=0$

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 6 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \textcircled{1} & 3 & 3 & 2 \\ 0 & 0 & \textcircled{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \triangleq U$$

1st pivot, 2nd pivot — echelon form

$x = (\textcircled{u}, v, \textcircled{w}, y)^T$

u, w: corresponding to pivots. basic variable.

v, y: free variables.

* Solutions can be expressed in terms of free variables.

$3w + y = 0 \Rightarrow w = -\frac{1}{3}y \quad \cdots (i)$

$u + 3v + 3w + 2y = 0$

$\Rightarrow u + 3v + y = 0$

$\Rightarrow u = -3v - y \quad \cdots (ii)$

From (i) and (ii)

$$\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v - y \\ v \\ -\frac{1}{3}y \\ y \end{bmatrix} = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

$N(A) = \{$ all the linear combinations of $(-3, 1, 0, 0)^T$ and $(-1, 0, -\frac{1}{3}, 1)^T \}$

Ex) $B = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix}$, $Bx = 0$?

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ \textcircled{v} \\ w \end{bmatrix}$$

v: free variable.

$w = 0$, $u + 2v = 0$, $u = -2v$

$\therefore N(B) = \{ v(-2, 1, 0),\ v \in R \}$

B. Inhomogeneous case. $Ax=b$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Start with

$$\begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 5 & b_2 \\ -1 & -3 & 3 & 0 & b_3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2-2b_1 \\ 0 & 0 & 6 & 2 & b_3+b_1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-2b_2+5b_1 \end{bmatrix}$$

If $b_3 - 2b_2 + 5b_1 \neq 0$, the system has no solution.

Otherwise, it has solution.

For example $(b_1, b_2, b_3)^T = (1, 5, 5)$

Then, the system

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ & & 3 & 1 \\ & & & \end{bmatrix}\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \end{bmatrix}$$

Again, v and y are free variables.

$$3w + y = 3$$

$$w = 1 - \frac{1}{3}y$$

$$u + 3v + 3w + 2y = 1$$

$$u + 3v + 3 + y = 1$$

$$u = -2 - 3v - y$$

$$\therefore \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2-3v-y \\ v \\ 1-\frac{1}{3}y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{particular solution}} + \underbrace{v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}}_{\in N(A)\text{ homogeous solution}}$$

$$\boxed{x_{general} = x_{particular} + x_{homogeneous}}$$

To summarize, after the elimination

A has r pivots and

$n-r$ free variables.

r is called the rank of A.

2.3 Linear Independence, Basis, and Dimension.

* Linear Independence

Ex) $(1, 0)$ and $(0, 1)$ linearly independent.

$(1, 2)$ and $(2, 4)$ dependent.

$(1, 1, 0), (0, 1, 1), (1, 0, 1)$ independent

$(1, 1, 0)\ (0, 1, 1)\ (1, 2, 1)$ dependent.

Def)
Vectors $v_1, v_2, \cdots, v_k$ are linearly independent,
if $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ implies
$c_1 = c_2 = \cdots = c_k = 0$.

Otherwise, they are linearly dependent.

Note) If $v_1, \cdots, v_k$ are dependent,
there exists $c_i \neq 0$ such that
$c_1 v_1 + \cdots + c_i v_i + \cdots c_k v_k = 0$.

Then $v_i = -\frac{c_1}{c_i} v_1 + \cdots + (-\frac{c_{i-1}}{c_i}) v_{i-1} + (-\frac{c_{i+1}}{c_i}) v_{i+1} + \cdots + (-\frac{c_k}{c_i}) v_k$.

In other words,
a vector is expressed by a combination of the other vectors.

Ex) • $v_1, v_2, \cdots, v_{k-1}, \bar{0}$ are dependent.

$\because\ 0v_1 + \cdots + 0v_{k-1} + 1 \cdot \bar{0} = 0$.

• Columns of

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$ are dependent.

$$\because \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

• Rows of A

$$\begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 6 & 2 & b_3 + b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_2 + 5b_1 \end{bmatrix}$$

(row 3) − 2 (row 2) + 5 (row 1) = $\bar{0}$

$\therefore$ Rows are dependent.

• Columns of

$$B = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$ are independent

$$\because\ c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 \to 0$ (3), $c_2 \to 0$ (2), $c_3 \to 0$ (1)

* Columns of A are independent

$\Leftrightarrow$ $Ax = 0$ has the only solution $x = 0$.

* $A = m\left[\right]$ (with n columns), $n > m$ i.e. more unknowns than equations.

The columns of A are dependent.

($\because$ After elimination, A has at most m pivots, at least $n-m$ free variables.

Therefore $Ax = 0$ has nonzero solutions) //

* ## Spanning

$w_1, \cdots, w_\ell$ span a space V,

if V consists of all linear combinations of $w_1, \cdots, w_\ell$. i.e., For every $v \in V$

$v = c_1 w_1 + \cdots + c_\ell w_\ell$ for some coefficients c_i's.

Ex). Column space is spanned by the column vectors.

- $\mathbb{R}^n$ is spaned by

$e_1 = (1, 0, \cdots, 0)^T$
$e_2 = (0, 1, \cdots, 0)^T$
$\vdots$
$e_n = (0, 0, \cdots, 1)^T$.

* ## Basis

A basis for a space V is a set of vectors

(i) which are linearly independent.

(ii) which span V.

Ex) $(1, 2)^T$ and $(1, 3)^T$ compose a basis for $\mathbb{R}^2$

$\because$ $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightarrow (c_1, c_2)^T = (0, 0)$ (independence)

For every $(x, y)^T$

$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

has a solution (c_1, c_2) (spanning)

Dimension

Any two bases for a space V have the same number of vectors

The number is called the dimension of V.

Proof) skipped. Try it yourself.

* Remarks

① Any linearly independent set in V can be extended to a basis by adding more vectors if necessary.

Ex) Find basis for $\mathbb{R}^3$, starting from

$\{(1, 2, 0)^T, (1, 3, 0)^T\}$

$\Rightarrow \{(1, 2, 0), (1, 3, 0), (0, 0, 1)\}$.

② Any spanning set in V can be reduced to a basis by discarding vectors if necessary.

$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\}$

2.4 Four Fundamental Spaces

1. Column space $R(A)$ $\subset \mathbb{R}^m$
2. Nullspace $N(A)$ $\subset \mathbb{R}^n$
3. Row space $R(A^T)$: $\subset \mathbb{R}^n$
   the space spanned by the rows of A
4. Left nullspace $N(A^T)$: $\subset \mathbb{R}^m$
   $\{y : A^T y = 0\}$ or equivalently $\{y : y^T A = 0\}$

* Row space $R(A^T)$

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} v_1 = w_1 \\ v_2 - 2v_1 = w_2 \\ v_3 - 2v_2 + 5v_1 = w_3 = 0 \end{matrix}$$

$$v_2 = w_2 + 2w_1$$
$$v_3 = 2v_2 - 5v_1$$
$$= 2w_2 + 2w_1 - 5w_1$$
$$= 2w_2 - 3w_1$$

$R(A^T) = R(U^T)$

$x \in R(A^T)$

$$x = c_1 v_1 + c_2 v_2 + c_3 v_3$$
$$= c_1 w_1 + c_2 (w_2 + 2w_1) + c_3 (2w_2 - 3w_1)$$
$$= d_1 w_1 + d_2 w_2$$

$\therefore \ x \in R(U^T)$

Similarly, if $x \in R(U^T)$, $x \in R(A^T)$.

This is natural, because Gaussian elimination is performed such that the rows of U are the combinations of the rows of A and the rows of A can be obtained from the rows of U.

A basis of $R(A^T) = \{(1,3,3,2), (0,0,3,1)\}$

The dimension of $R(A^T) = 2 =$ the rank of A.

$= \#$ of pivots in U

* Nullspace $\mathcal{N}(A)$

Note that $\mathcal{N}(A) = \mathcal{N}(U)$, because

$Ax = 0$ and $Ux = 0$ has the same solutions.

$$Ax = 0 \rightarrow x = v\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

A basis for $\mathcal{N}(A)$

$= \{(-3, 1, 0, 0), (-1, 0, -\frac{1}{3}, 1)\}$

The dimension of $\mathcal{N}(A)$

$= 2 =$ # of free variables in U

$= n - r$

* Column space $R(A)$

Note that $R(A) \neq R(U)$.

But $Ax = 0$ and $Ux = 0$ have the same solutions

$\Rightarrow$ The dependency of the columns of A

= 〃 〃 of U

$\Rightarrow$ A basis for $R(A)$ is composed of the columns of A corresponding to the pivot elements.

$\Rightarrow$ The dimension of $R(A)$ = # of pivots

= rank of A

= The dimension of $R(A^T)$.

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① $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$ $\quad U = \begin{bmatrix} ① & 3 & 3 & 2 \\ 0 & 0 & ③ & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In U, 2nd column = $3 \times$ (1st column)

$\therefore$ $(3, -1, 0, 0)^T$ is a solution of $Ux = 0$

〃 is a solution of $Ax = 0$

$\therefore$ In A, 2nd column = $3 \times$ (1st column)

In U, 4th column

= (1st column) + $\frac{1}{3}$ (3rd column)

$\therefore$ $(1, 0, \frac{1}{3}, -1)$ is a solution of $Ux = 0$

〃 of $Ax = 0$

$\therefore$

$\therefore$ In A, 4th column

= (1st column) + $\frac{1}{3}$ (3rd column)

In U, 1st and 3rd columns are independent.

$U \begin{bmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{bmatrix} = 0$ has the only solution $\alpha = \beta = 0$.

$A \begin{bmatrix} \alpha \\ 0 \\ \beta \\ 0 \end{bmatrix} = 0$ 〃

In A, 1st and 3rd columns are independent.

A basis for $R(U) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow$ 〃 $R(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 3 \end{bmatrix} \right\}$

$\dim R(A) = 2$ = # of pivots in U.

* Left Nullspace $N(A^T)$.

Recall that $N(A)$ has the dimension $n - r$ (column dim = row dim)

Therefore $N(A^T)$ has the dimension $m - r$

$$y^T A = 0$$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} \longrightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow v_3 - 2v_2 + 5v_1$$

$\therefore$ $v_3 - 2v_2 + 5v_1 = 0$

$$y^T = (5, -2, 1)$$

$$y^T A = 5v_1 - 2v_2 + v_3 = 0.$$

$\therefore$ $N(A^T) = \{ c(5, -2, 1), c \in R \}$

$\dim N(A^T) = 1 = 3 - 2.$

To summarize, we have

Fundamental Thm for Linear Algebra, Part I

1. $R(A)$ = column space of A, $\dim = r$
2. $N(A)$ = nullspace of A, $\dim = n - r$
3. $R(A^T)$ = row space of A, $\dim = r$
4. $N(A^T)$ = left nullspace of A, $\dim = m - r$.

Linear Transformations

A : $m \times n$ matrix

$\Rightarrow$ A is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$
transformation

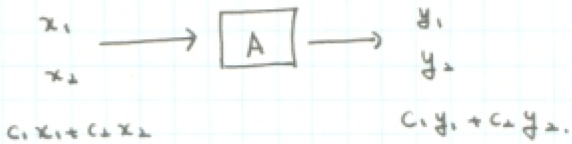
ex)

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$(1,1,1)^T$ is mapped to $(3,4)^T$ by the function $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

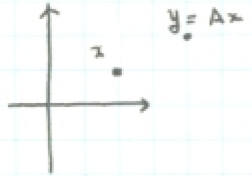
$\Rightarrow$ A is a linear transformation:

$$A(c_1 x_1 + c_2 x_2) = c_1 A x_1 + c_2 A x_2$$

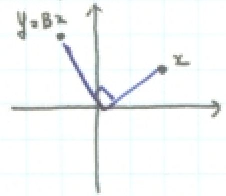


Ex)

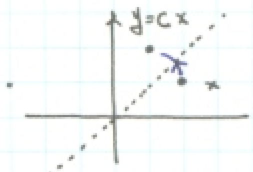
$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ stretching



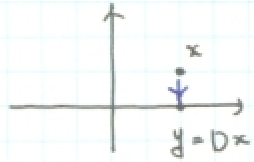
$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}$ rotation



$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection



$D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projection



Matrices are not the only linear transforms.

Let f be a function from a vector space V to another vector space W.

$$f: V \to W.$$

f is called a linear transform if

$$f(c_1 x_1 + c_2 x_2) = c_1 f(x_1) + c_2 f(x_2)$$

for all $c_1, c_2 \in R$ and $x_1, x_2 \in V$

Ex) P_n : the set of all polynomials in t whose order is less than or equal to n.

$$x \in P_n \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow y \in P_n$$

$$x(t) = a_0 + a_1 t + \cdots + a_n t^n$$

$$\downarrow$$

$$y(t) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

Differentiation is linear, since

$$\frac{d}{dt}\left(c_1 x_1(t) + c_2 x_2(t)\right) = c_1 \frac{d}{dt} x_1(t) + c_2 \frac{d}{dt} x_2(t).$$

But, every linear transformation can be represented by a matrix.

$x(t) = a_0 + a_1 t + \cdots + a_n t^n$

$\overset{\text{represent}}{\Longrightarrow} (a_0, a_1, \cdots, a_n)^T$

Then, the above differentiation can be represented by a $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

because $(a_0, a_1, \cdots, a_n)$ is mapped to $(a_1, 2a_2, \cdots, n a_n, 0)$

$$\frac{d}{dt}: P_3 \to P_3$$

$$x(t) = 2 + t + t^2 - 2t^3 \to y(t) = 1 + 2t - 6t^2$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -6 \\ 0 \end{bmatrix}$$

$R(A) = \{(\alpha, \beta, \gamma, 0)^T : \alpha, \beta, \gamma \in \mathbb{R}\} = P_2$

$N(A) = \{(c, 0, 0, 0)^T : c \in \mathbb{R}\}$ = the set of constant functions

A few linear transforms

(1) Rotation matrix

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

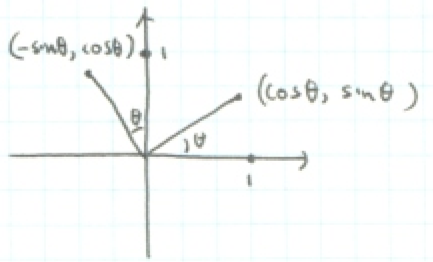
Note that the first column $(\cos\theta, \sin\theta)^T$ is obtained by rotating $(1,0)^T$

the second column $(-\sin\theta, \cos\theta)^T$ is obtained by rotating $(0,1)^T$

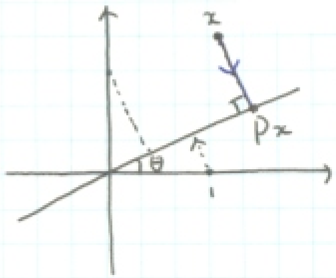
This is natural because

the first column $= Q_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

the second column $= Q_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



(2) Projection onto θ-line



$$P \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \times \cos\theta \\ \cos\theta \times \sin\theta \end{bmatrix}$$

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\theta \\ \sin\theta \sin\theta \end{bmatrix}$$

$$\Rightarrow \quad P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Note that $P^2 = P$.

$$\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix} = \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix}$$

$$= (c^2 + s^2) \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

(3) Try "Reflection with respect to the θ-line" yourself.