Matrix - Basics

$$A = \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$a_{23} = 16 =$ entry in row 2 and column 3

$$= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \qquad c_1 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \quad c_2 = \begin{bmatrix} 1 \\ -0.2 \end{bmatrix} \quad c_3 = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} r_1^T \\ r_2^T \end{bmatrix} \qquad r_1 = \begin{bmatrix} 0.3 \\ 1 \\ -5 \end{bmatrix} \quad r_2 = \begin{bmatrix} 0 \\ -0.2 \\ 16 \end{bmatrix}$$

Notations: By default, all vectors are column vectors.

Transpose

$$A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} = \begin{bmatrix} 0.3 & 0 \\ 1 & -0.2 \\ -5 & 16 \end{bmatrix}$$

In general,

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & & a_{mn} \end{bmatrix}$$

- $m \times n$ matrix
- if $m = n$, A is a square matrix

$B = A^T$

- $n \times m$ matrix
- $b_{ij} = a_{ji}$

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

$b_{21} = 2 = a_{12}$

$A = B$

① A and B have the same size

② $a_{ij} = b_{ij}$ for all i and j

$C = A + B$

① addition is defined only if A and B have the same size

② $c_{ij} = a_{ij} + b_{ij}$

ex) $A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

$$C = A + B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

Scalar multiplication

$$B = cA$$

① $b_{ij} = c\,a_{ij}$

ex) $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Rightarrow 2A = \begin{bmatrix} 2 & 8 \\ 4 & 10 \\ 6 & 12 \end{bmatrix}$

Some rules

- $A + B = B + A$ commutativity
- $(A + B) + C = A + (B + C) = A + B + C.$ associativity
- $A + O = A$ where $O = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$
- $A + (-A) = O$
- $c(A + B) = cA + cB$
- $(c + k)A = cA + kA$
- $c(kA) = (ck)A = ckA$
- $1A = A$

Matrix multiplication

$$\underset{m \times k}{C} = \underset{m \times n}{A} \; \underset{n \times k}{B}$$

ex)

$$\begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 18 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$\text{row}^T \cdot \text{column}$

$$\begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix} \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} r_1^T c_1 & r_1^T c_2 & r_1^T c_3 & r_1^T c_4 \\ r_2^T c_1 & r_2^T c_2 & r_2^T c_3 & r_2^T c_4 \\ r_3^T c_1 & r_3^T c_2 & r_3^T c_3 & r_3^T c_4 \end{bmatrix}$$

$r_i^T c_j$ = inner product of r_i and c_j

$= r_i \circ c_j$

$$c_{ij} = r_i^T c_j$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

For example.

$c_{21} = 18 = 4 \times 2 + 0 \times 5 + 2 \times 9$

$= a_{21} b_{11} + a_{22} b_{21} + a_{23} b_{31}$

ex)

$$\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 18 \\ 43 \end{bmatrix}$$

2×1 matrix = column vector

ex)

$$\begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

row vector × column vector = scaler

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

In general $AB \neq BA$

$$\underbrace{m \times n \;\; n \times m}_{m \times n} \qquad \underbrace{n \times m \;\; m \times n}_{n \times n}$$

ex)

$$\begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 100 & 100 \end{bmatrix} = \begin{bmatrix} 99 & 99 \\ -99 & -99 \end{bmatrix}$$

Some rules

- $(kA)B = k(AB) = A(kB) = kAB$
- $A(BC) = (AB)C$
- $(A+B)C = AC + BC$
- $C(A+B) = CA + CB$

Proof) $(A+B)C = AC + BC$

$(A+B)C = D$, $AC + BC = F$

$$d_{ij} = \sum_k (a_{ik} + b_{ik}) c_{kj}$$

$$= \sum_k a_{ik} c_{kj} + \sum_k b_{ik} c_{kj}$$

$$= F$$

Proof) $A(BC) = (AB)C$

$BC = D$ $AB = F$

$$[AD]_{ij} = \sum_k a_{ik} d_{kj}$$

$$= \sum_k a_{ik} \sum_l b_{kl} c_{lj}$$

$$= \sum_k \sum_l a_{ik} b_{kl} c_{lj}$$

$$= \sum_l \left(\sum_k a_{ik} b_{kl} \right) c_{lj} = \sum_l f_{il} c_{lj}$$

$$[FC]_{ij}$$

Matrix multiplication is a linear transform

$y = f(x)$

f is linear

① $f(x_1 + x_2) = f(x_1) + f(x_2)$ additivity

② $f(c x_1) = c f(x_1)$ homogeneity

$y = A x$

$A(x_1 + x_2) = A x_1 + A x_2$ additivity

$A(c x_1) = c(A x_1)$ homogeneity

Transition properties

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(c A)^T = c A^T$
- $(A B)^T = B^T A^T$

Special matrices

- $A^T = A$ symmetric matrix

- $\begin{bmatrix} 1 & 4 & 2 \\ & 3 & 2 \\ & & 6 \end{bmatrix}$ upper triangular matrix

- $\begin{bmatrix} 1 & & \\ 4 & 3 & \\ 2 & 2 & 6 \end{bmatrix}$ lower triangular matrix

- $I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ identity matrix

  $Ix = x$ for all x.