Linear Algebra

Chap. 1 Matrices and Gaussian Elimination

* Gaussian Elimination : an example.

Suppose that we are given a system of equations

forward elimination

$$\begin{cases} 2u + v + w = 5 & \cdots ① \\ 4u - 6v = -2 & \cdots ② \\ -2u + 7v + 2w = 9 & \cdots ③ \end{cases}$$

⇓ eliminating u from ② and ③

$$\begin{cases} 2u + v + w = 5 & \cdots ① \\ -8v - 2w = -12 & \cdots ②' \quad (= ② - 2 \times ①) \\ 8v + 3w = 14 & \cdots ③' \quad (= ③ + ①) \end{cases}$$

⇓ eliminating v from ③'

$$\begin{cases} 2u + v + w = 5 & \cdots ① \\ -8v - 2w = -12 & \cdots ②' \\ w = 2 & \cdots ③'' \quad (= ③' + ②') \end{cases}$$

back substitution.

$$-8v = -12 + 2w = -8 \Rightarrow v = 1$$

$$2u = 5 - v - w = 2 \Rightarrow u = 1.$$

Therefore, we have the solution

$$u = 1, \quad v = 1, \quad w = 2$$

The same system can be written as

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix},$$

and the elimination can be simply denoted by

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc:c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

upper triangle

pivot elements

Note 1. Row exchanges are necessary in some cases.

$$u + v + w = 3$$
$$2u + 2v + 5w = 9$$
$$4u + 6v + 8w = 18$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 9 \\ 4 & 6 & 8 & 18 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{row exchange}} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

After back-substitution,

$$w = 1$$
$$2v = 6 - 4w = 2 \Rightarrow v = 1$$
$$u = 3 - v - w = 1$$

Note 2. Some systems break down (singular case)

$$u + v + w = 3$$
$$2u + 2v + 5w = 9$$
$$4u + 4v + 8w = 16\ (17)$$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 5 & 9 \\ 4 & 4 & 8 & 16\,(17) \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 4 & 4\,(5) \end{bmatrix} \rightarrow \begin{bmatrix} \textcircled{1} & 1 & 1 & 3 \\ 0 & 0 & \textcircled{3} & 3 \\ 0 & 0 & 0 & 0\,(1) \end{bmatrix}$$

Only two pivots.

$$w = 1$$
$$u = 3 - v - w = 2 - v.$$

$\therefore (u, v, w) = (2-\alpha, \alpha, 1)$

There are infinitely many solution) } singular cases.

$0 = 1$ (inconsistent)

There is no solution

* Cost of elimination

For simplicity, ignore the right hand sides.

$$n \left\{ \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \right. \quad (\text{width } n)$$

From 1st pivot, we need $n^2 - n$ operations ($\times$ and $-$)

From 2nd pivot, we need $(n-1)^2 - (n-1)$ operations.

$\vdots$

In total, we need

$$\sum_{k=1}^{n} (k^2 - k) \approx \frac{1}{3} n^3 = O(n^3)$$

* Cost of back-substitution

From the last row, 1 op

2nd last row, 2 op

$\vdots$

1st row n op

In total, we need

$$\sum_{k=1}^{n} k \approx \frac{1}{2} n^2 = O(n^2).$$

Two basic operations in Gaussian elimination

① Subtract a multiple l of row j from row i.

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

subtract $2 \times$ (row 1) from row 2

This can be represented by a matrix multiplication.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

Or more compactly,

$$\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}$$

In general, E_{ij} denotes the **elementary matrix**, which subtracts a multiple l of row j from row i.

It has 1 on the diagonal
$-l$ in row i, column j
0 otherwise.

Ex) $E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -l & 0 & 0 & 1 \end{bmatrix}$

② Exchange row i with row j

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} \xrightarrow[\text{row 2 with row 3}]{\text{exchange}} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

This can also be represented by a matrix multiplication

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 3 & 3 \end{bmatrix}.$$

In general, P_{ij} denotes the **permutation matrix**, which exchanges row i with row j

It is obtained by exchanging rows i and j of the identity matrix

Ex) $P_{23} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Inverses of elementary matrix and permutation matrix.

① $E_{41}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\ell & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \ell & 0 & 0 & 1 \end{bmatrix}$

Inverse of (subtracting $\ell \times$ row 1 from row 4)

= Adding $\ell \times$ row 1 to row 4.

② $P_{23}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P_{23}$

Inverse of (exchanging rows i and j)

= Exchanging rows i and j.

Triangular Factorization $A = LU$

Let's consider the first example of Gaussian elimination, where row exchanges are not necessary.

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b.$$

Ignoring the right-hand sides, we observe

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\Downarrow$$

$$A^{(1)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix} = E_\alpha A \quad \text{where } E_\alpha = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$A^{(2)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{bmatrix} = E_\beta A^{(1)} \quad \text{where } E_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$A^{(3)} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} = E_\gamma A^{(2)} \quad \text{where } E_\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A^{(3)} = U \quad \text{(upper triangular matrix)}$$

$$= E_\gamma E_\beta E_\alpha A.$$

$$\Rightarrow \quad A = (E_\gamma E_\beta E_\alpha)^{-1} U$$

$$= E_\alpha^{-1} E_\beta^{-1} E_\gamma^{-1} U$$

Note that

$$E_\alpha^{-1} E_\beta^{-1} E_\gamma^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix}$$

$$= L \quad \text{(lower triangular matrix)}$$

Therefore

$$A = LU.$$

The factorization makes the computation of

$$Ax = b \text{ easier.}$$

$$LUx = b$$

$Ux \stackrel{\triangle}{=} c \quad \Rightarrow \quad$ $\boxed{Lc = b}$

$$\begin{bmatrix} 1 & & \\ 2 & 1 & \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$c_1 = 5, \quad c_2 = -2 - 2c_1 = -12$ // Solve triangular system 1

$c_3 = 9 + c_1 + c_2 = 2.$

$\boxed{Ux = c}$

$$\begin{bmatrix} 2 & 1 & 1 \\ & -8 & -2 \\ & & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

$w = 2, \quad -8v - 4 = -12 \rightarrow v = 1$ // Solve triangular system 2

$2u + 1 + 2 = 5 \rightarrow u = 1$

Remarks

(a) The above example shows how we solve $Ax = b$ in practice, i.e., via Gaussian elimination or LU factorization. We donot first compute A^{-1} and then $x = A^{-1}b$, which is much more complicated.

(b) If row exchanges are necessary, we carry out them first to obtain $A' = PA$ and obtain the factorization

$$PA = LU$$

Ex) $PA = LU$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$$P_{12}A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{E_{31}^{(-2)}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

$$\xrightarrow{E_{32}^{(-3)}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad E_{32}^{(-3)} E_{31}^{(-2)} P_{12} A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$P_{12}A = \left[E_{31}^{(-2)}\right]^{-1}\left[E_{32}^{(-3)}\right]^{-1} U$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ 2 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 3 & 1 \end{bmatrix} U$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ 2 & 3 & 1 \end{bmatrix} U$$

$$\therefore \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Gauss - Jordan Method to find A^{-1}

* If a matrix A is nonsingular (invertible), we can make it diagonal using elementary matrices and permutation matrices. For example

$$E_4 P_2 E_3 P_1 E_2 E_1 A = D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}$$

Then,

$$D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 A = I.$$

Therefore

$$A^{-1} = D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 .$$

If we start the forward elimination with

$$[\,A \quad I\,],$$

the result is

$$D^{-1} E_4 P_2 E_3 P_1 E_2 E_1 [\,A \quad I\,]$$

$$= [\,I \quad A^{-1}\,].$$

To summarize, carry out the elimination with $[A \;\; I]$ to make the identity in the left side, i.e, $[I \;\; B]$.

Then B is the inverse of A.

Example) $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 0 & \frac{3}{4} & \frac{3}{4} & \frac{1}{8} & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 2 & 0 & 0 & \frac{3}{2} & -\frac{5}{8} & -\frac{3}{4} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}$$

Inverses

A matrix A is invertible if there exists B such that $BA = I$ and $AB = I$. There is at most one such B, called the inverse of A and denoted by A^{-1}.

* $BA = I,\ AC = I \Rightarrow B = C$

  $(\because\ B = B(AC) = (BA)C = C)$

* $(A^{-1})^{-1} = A$

* $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{bmatrix}$

* $(AB)^{-1} = B^{-1}A^{-1}$

  $(\because\ (AB)B^{-1}A^{-1} = ABB^{-1}A^{-1} = AA^{-1} = I$

  $B^{-1}A^{-1}(AB) = B^{-1}A^{-1}AB = BB^{-1} = I$

Transposes

$A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}$

Def) $(A^T)_{ij} = A_{ji}$

* $(AB)^T = B^T A^T$

* $(A^{-1})^T = (A^T)^{-1}$

  $\left[\because\ (AA^{-1})^T = (A^{-1})^T A^T = I \Rightarrow (A^T)^{-1} = (A^{-1})^T\right]$