Signals and Systems

Linear Time-Invariant (LTI) Systems

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Discrete-Time LTI Systems

Representing Signals in Terms of Impulses

- Sifting property

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$= \cdots$$

$$+x[-2]\delta[n+2]$$

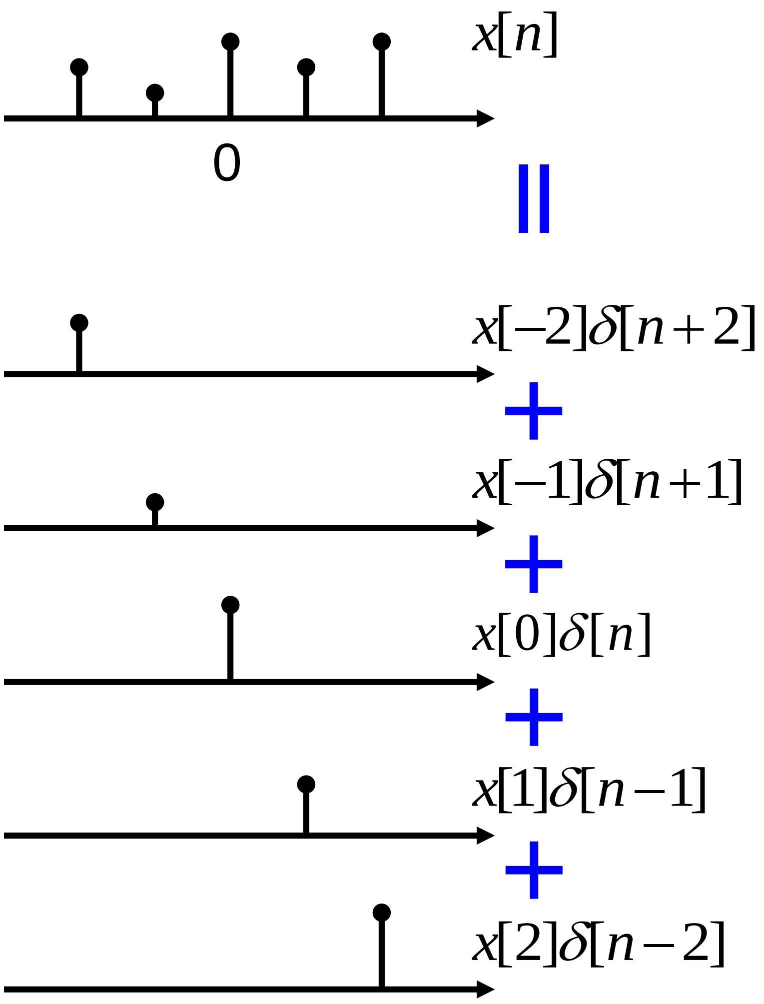
$$+x[-1]\delta[n+1]$$

$$+x[0]\delta[n]$$

$$+x[1]\delta[n-1]$$

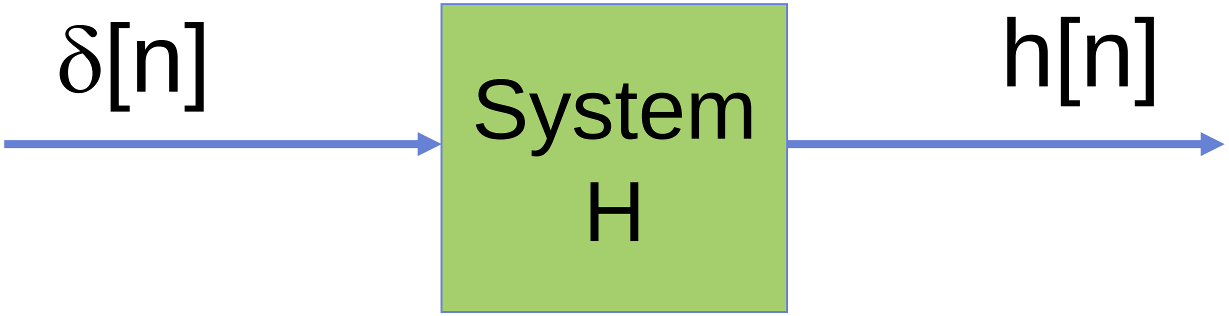
$$+x[2]\delta[n-2]$$

$$+\cdots$$



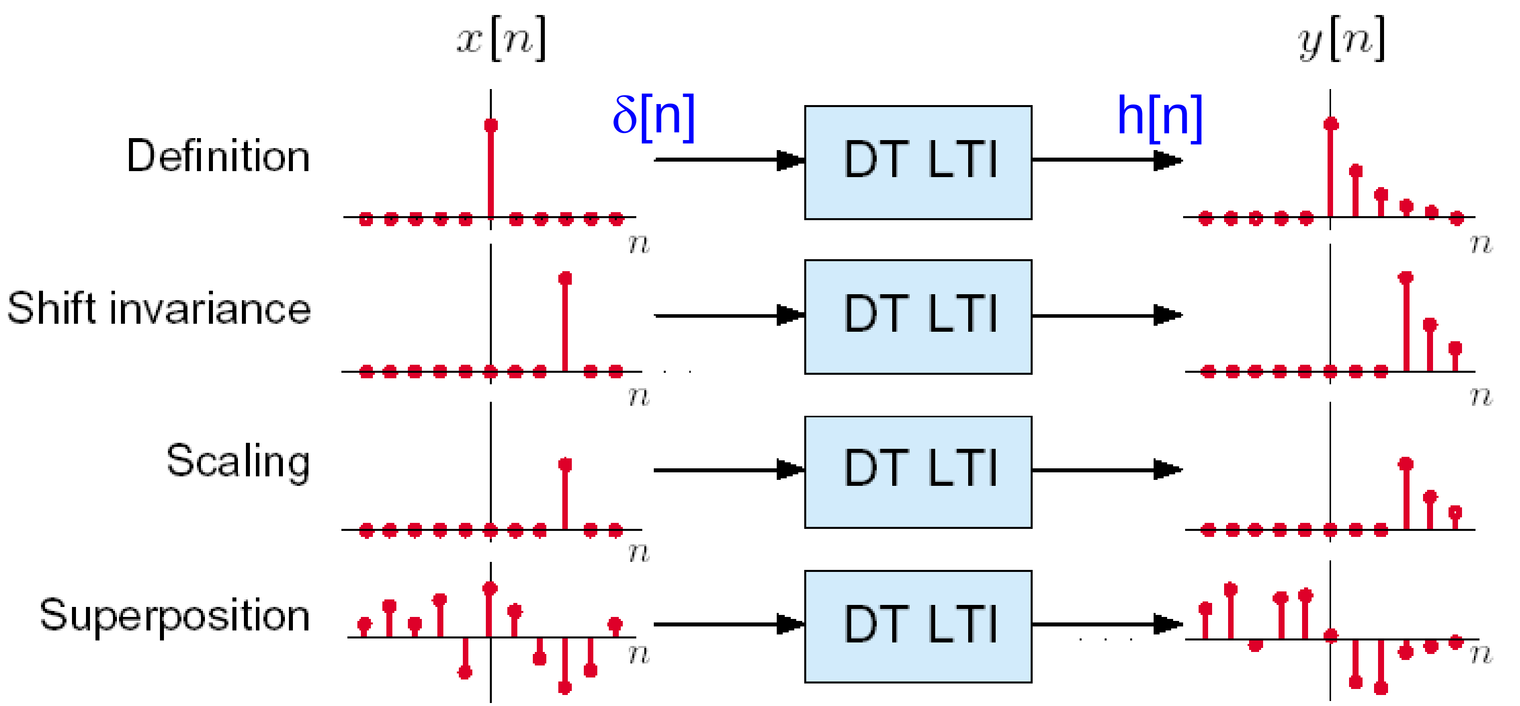
Impulse Response

- The response of a system H to the unit impulse $\delta[n]$ is called the impulse response, which is denoted by h[n]
  - $h[n] = H[\delta[n]]$



Convolution Sum

- Let h[n] be the impulse response of an LTI system.
- Given h[n], we can compute the response y[n] of the system to any input signal x[n].



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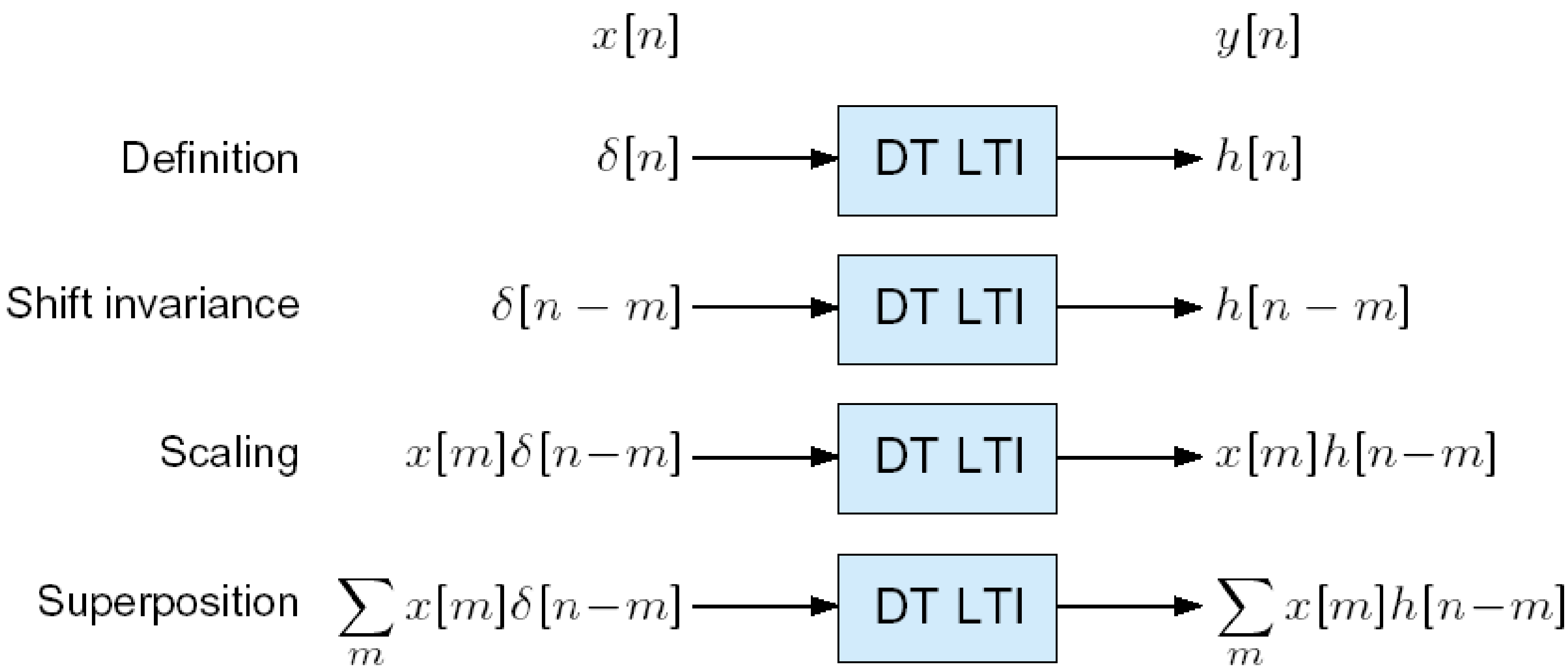


Convolution Sum

- Let h[n] be the impulse response of an LTI system.
- Given h[n], we can compute the response y[n] of the system to any input signal x[n].

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\begin{aligned} y[n] &= H[x[n]] \\ &= H\left[\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right] \\ &= \sum_{k=-\infty}^{\infty} H\left[x[k]\delta[n-k]\right] \\ &= \sum_{k=-\infty}^{\infty} x[k]H\left[\delta[n-k]\right] \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \end{aligned}$$

Convolution Sum

- Notation for convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- The characteristic of an LTI system is completely determined by its impulse response.

Convolution Sum

- To compute the convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Step 1 Plot x and h vs k since the convolution sum is on k.

Step 2 Flip $h[k]$ around the vertical axis to obtain $h[-k]$.

Step 3 Shift $h[-k]$ by n to obtain $h[n-k]$.

Step 4 Multiply to obtain $x[k]h[n-k]$.

Step 5 Sum on k to compute $\sum x[k]h[n-k]$.

Step 6 Change n and repeat **Steps 3-6**.

Example

- Consider an LTI system that has an impulse response $h[n] = \mathrm{u}[n]$

- What is the response when an input signal is given by

$$x[n] = \alpha^n u[n]$$

where $0<\alpha<1$?

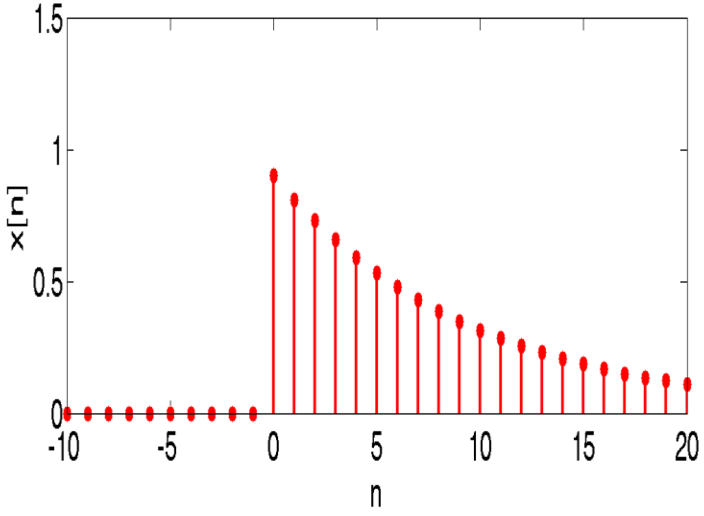
- For $n \geq 0$,

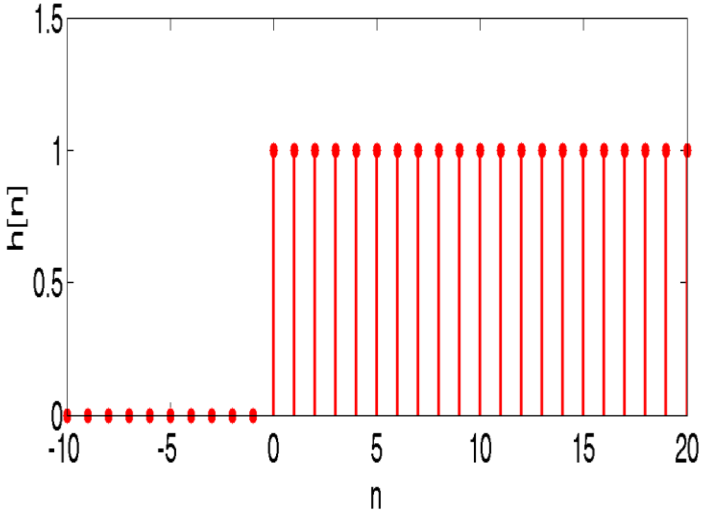
$$y[n] = \sum_{k=0}^{n} \alpha^k$$

$$= \frac{1-\alpha^{n+1}}{1-\alpha}$$

- Therefore,

$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$





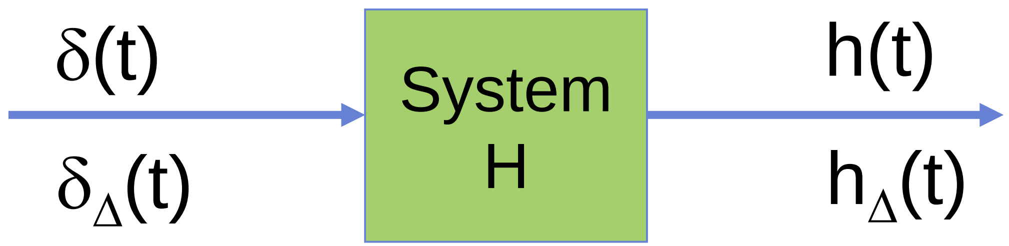
Convolution Sum

Demonstration

Continuous-Time LTI Systems

Impulse Response

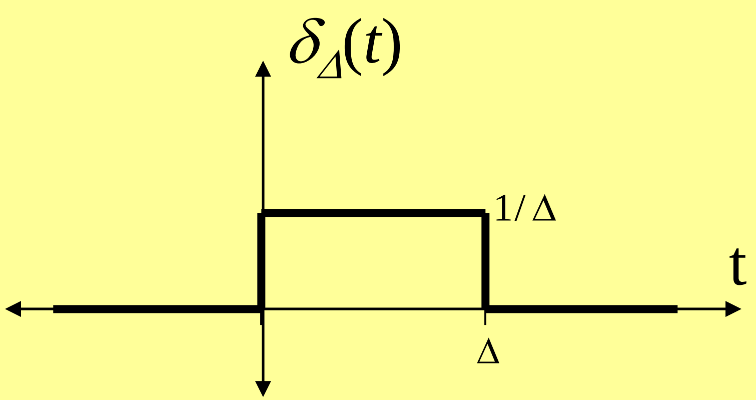
- The response of a system H to the unit impulse $\delta(t)$ is called the impulse response, which is denoted by $h(t)$
  - $h(t) = H(\delta(t))$



  - As $\Delta \to 0$, $\delta_\Delta(t) \to \delta(t)$ and $h_\Delta(t) \to h(t)$

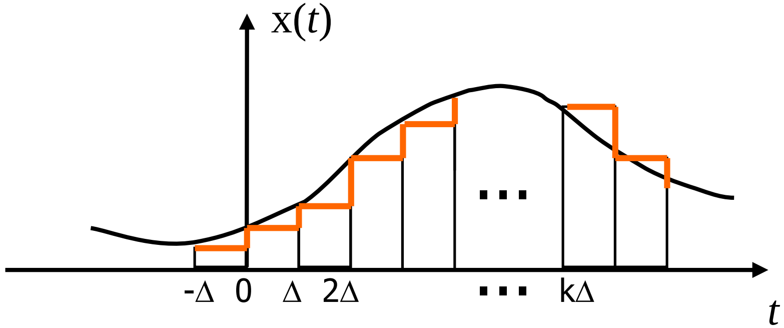
Recall the definition of approximated impulse function

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

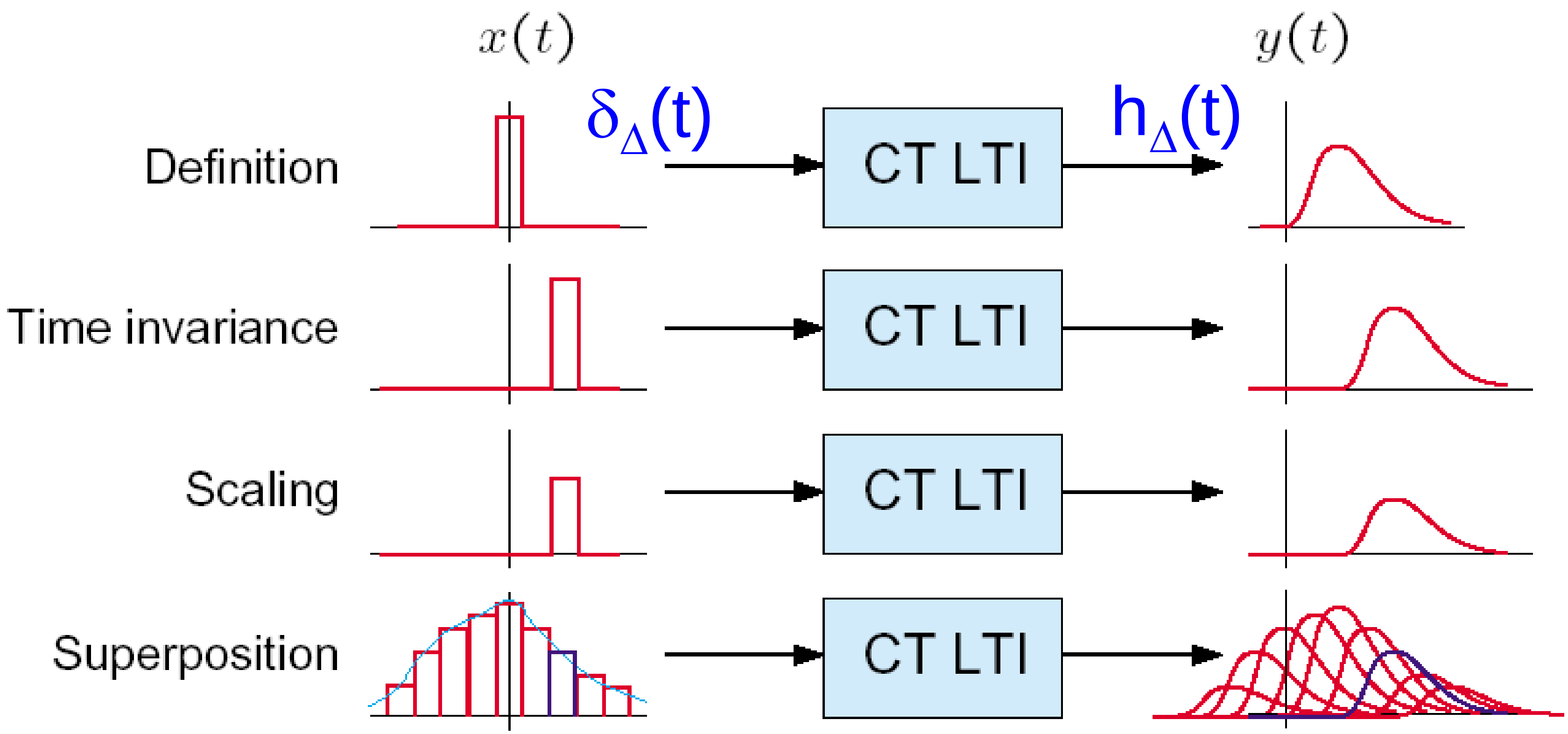


Staircase Approximation of x(t)

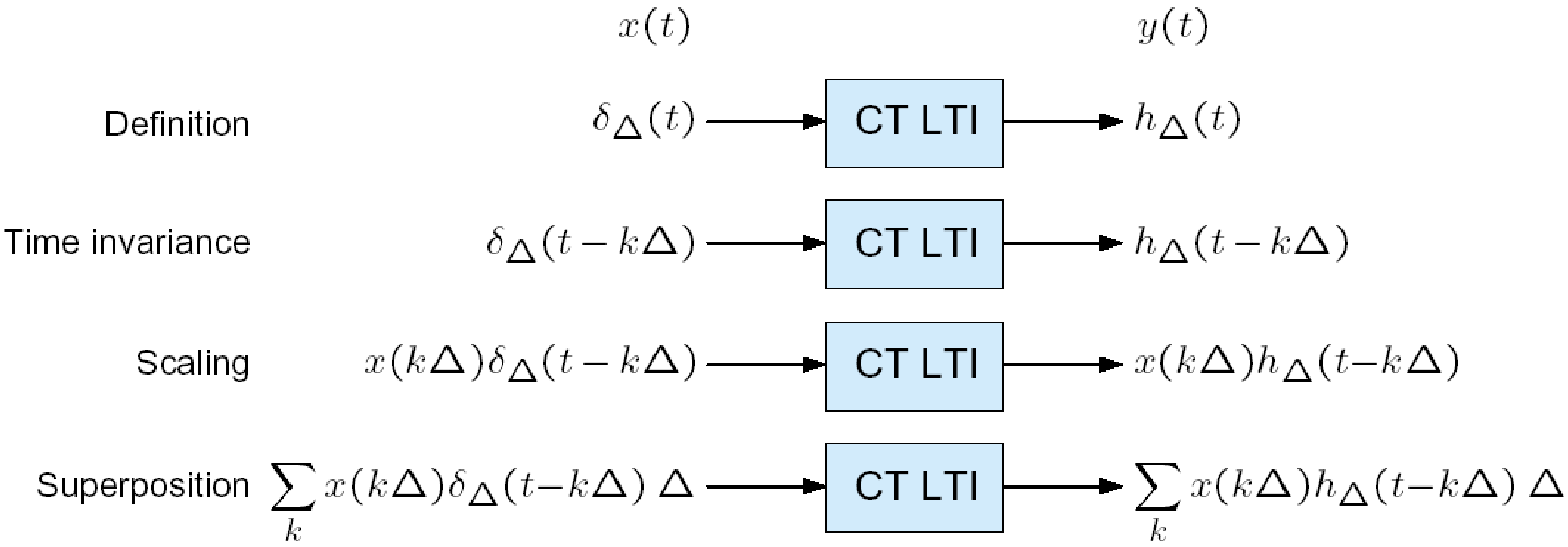
$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$



Convolution Integral



Convolution Integral



Convolution Integral

The derivation shows that a staircase approximation to the input

$$x_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

yields an approximation to the output

$$y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta)\Delta$$

Now we take the limit. As $\Delta \to 0$, ™$_\Delta$(t) $\to$ δ(t), $h_\Delta(t) \to h(t)$, $x_\Delta(t) \to x(t)$, and $y_\Delta(t) \to y(t)$. Also, the sums approach the integrals

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau$$ Sifting property

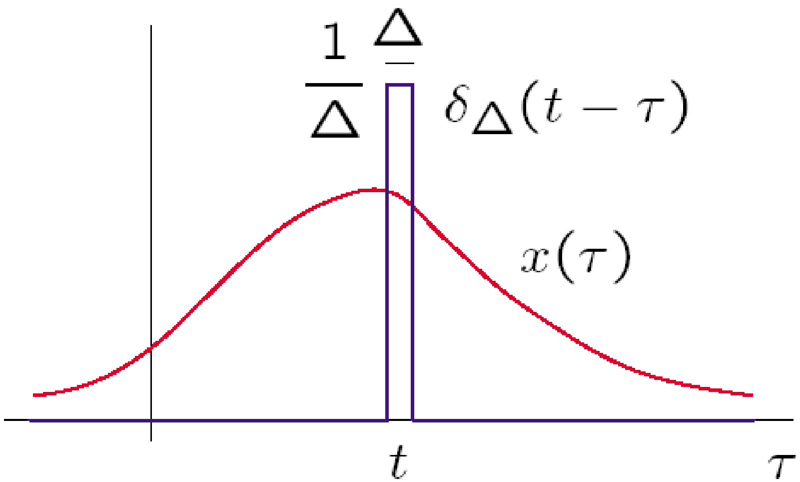
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$ Convolution integral

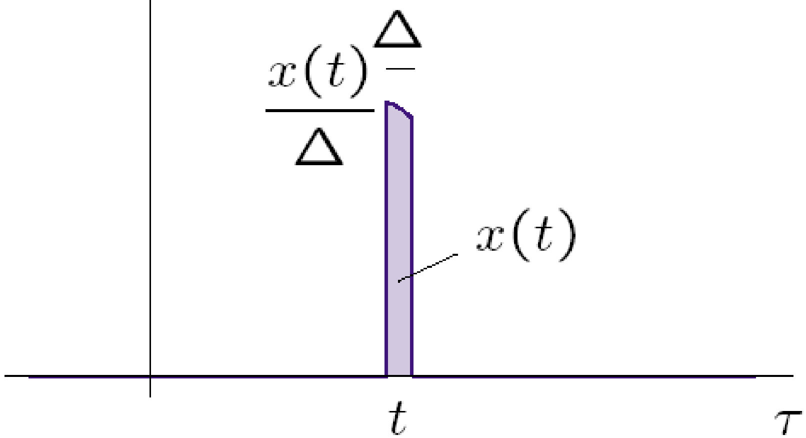
Another Interpretation of Sifting Property

To see the meaning of the sifting property

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

we approximate the impulse with a tall, narrow pulse $\delta_{\Delta}(t-\tau)$





Convolution Integral

- To compute the convolution integral

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Step 1 Plot x and h vs τ since the convolution integral is on τ.

Step 2 Flip $h(\tau)$ around the vertical axis to obtain $h(-\tau)$.

Step 3 Shift $h(-\tau)$ by t to obtain $h(t-\tau)$.

Step 4 Multiply to obtain $\mathrm{x}(\tau)\, h(t-\tau)$.

Step 5 Integrate on τ to compute $\int \mathrm{x}(\tau)\, h(t-\tau)\, d\tau$.

Step 6 Increase t and repeat **Steps 3-6**.

Example 1

- Let $x(t)$ be the input to a LTI system with unit impulse response $h(t)$

$$x(t) = e^{-at}u(t) \qquad a > 0$$

$$h(t) = u(t)$$

- For $t > 0$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau} & 0 < \tau < t \\ 0 & \text{otherwise} \end{cases}$$

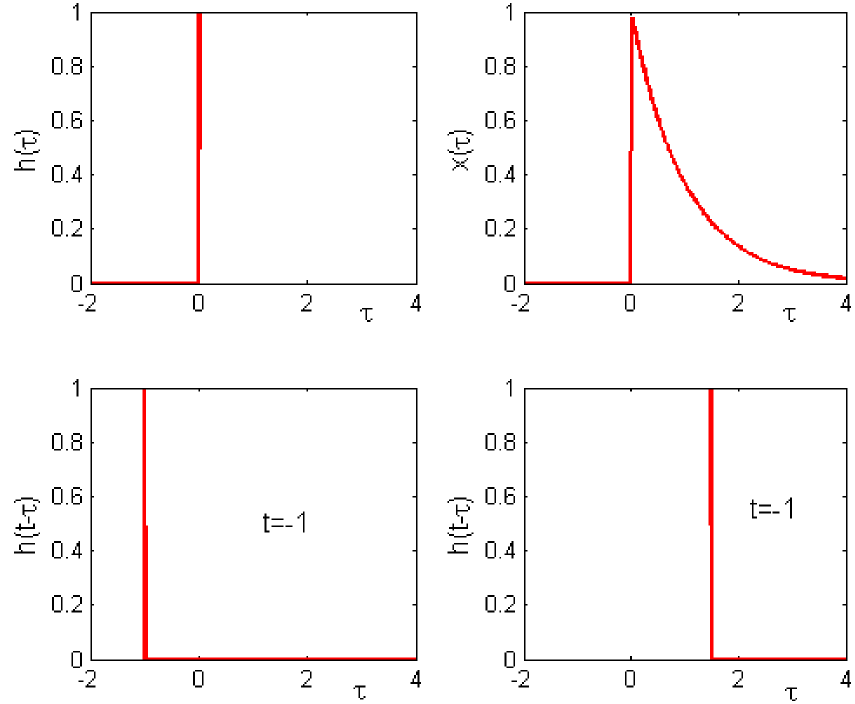
- We can compute $y(t)$ for $t > 0$

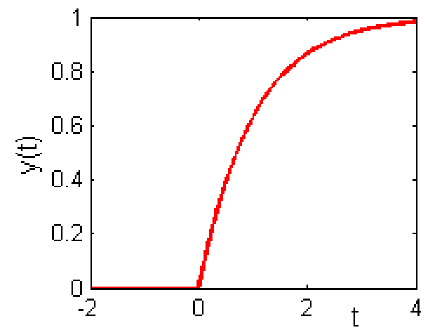
$$y(t) = \int_0^t e^{-a\tau}d\tau = -\tfrac{1}{a}e^{-a\tau}\Big|_0^t$$

$$= \tfrac{1}{a}\left(1 - e^{-at}\right)$$

- So for all t

$$y(t) = \tfrac{1}{a}\left(1 - e^{-at}\right)u(t)$$





In this example $a=1$

Example 2

- Calculate the convolution of the following signals

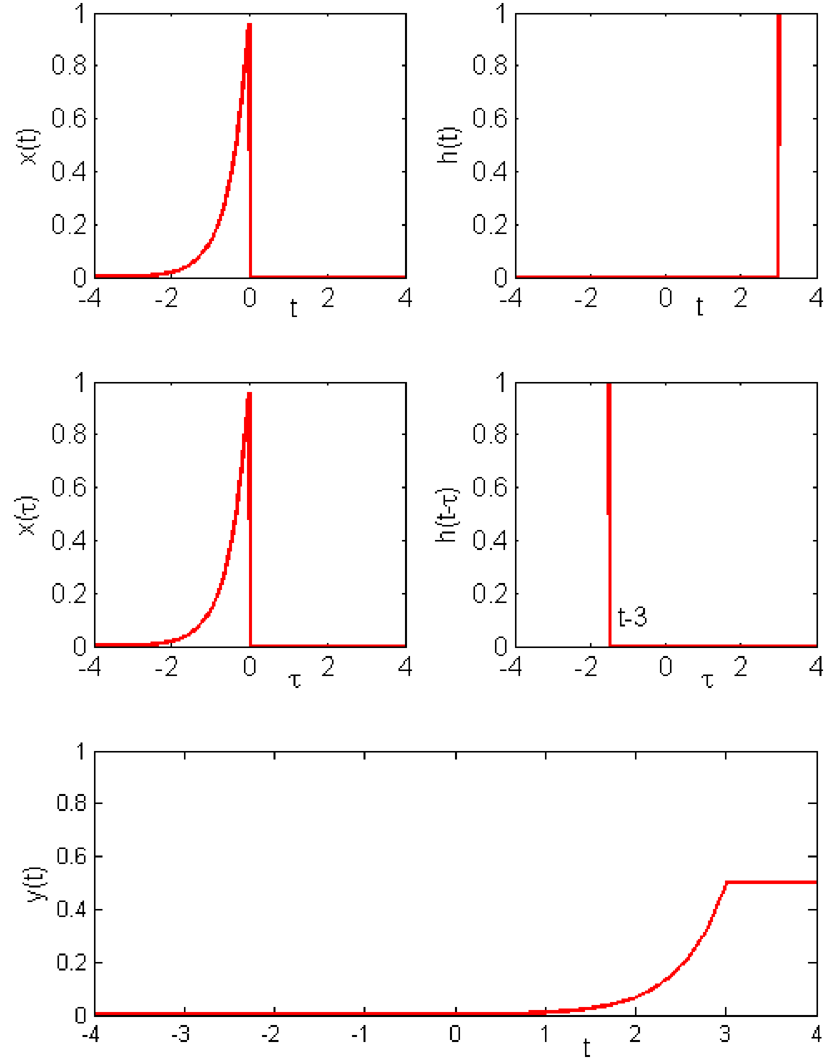
$$x(t) = e^{2t}u(-t)$$
$$h(t) = u(t-3)$$

- For t<3, the convolution integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \tfrac{1}{2} e^{2(t-3)}$$

- For t-3≥0, the product $x(\tau)h(t-\tau)$ is non-zero for $-\infty<\tau<0$, so the convolution integral becomes

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \tfrac{1}{2}$$



Properties of LTI Systems

Properties of Convolution

$$x[n]*y[n]=\sum_{k=-\infty}^{\infty}x[k]y[n-k]$$

$$x(t)*y(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)d\tau$$

- Commutative

  $x[n]*y[n]=y[n]*x[n]$

  $x(t)*y(t)=y(t)*x(t)$

- Distributive

  $x[n]*(y_1[n] + y_2[n])=x[n]*y_1[n] + x[n]*y_2[n]$

  $x(t)*(y_1(t) + y_2(t))=x(t)*y_1(t) + x(t)*y_2(t)$

- Associative

  $x[n]*(y_1[n]*y_2[n])=(x[n]*y_1[n])*y_2[n]$

  $x(t)*(y_1(t)*y_2(t))=(x(t)*y_1(t))*y_2(t)$

Causality of LTI Systems

- A system is causal if its output depends only on the past and the present values of the input signal.

- Consider the following for a causal DT LTI system:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

  - Because of causality h[n-k] must be zero for k>n.
  - In other words, h[n]=0 for n<0.

- Similarly for a CT LTI system to be causal
  - h(t) = 0 for t<0.

Causality of LTI Systems

- So the convolution sum for a causal LTI system becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

- Similarly, the convolution integral for a causal LTI system becomes

$$y[n] = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- So, if a given system is causal, one can infer that its impulse response is zero for negative time values, and use the above simpler convolution formulas.

Stability of LTI Systems

- A system is stable if a bounded input yields a bounded output (BIBO). In other words, if $|x[n]| < k_1$ then $|y[n]| < k_2$.

- Note that

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k]h[k] \right| \leq \sum_{k=-\infty}^{\infty} |x[n-k]||h[k]| \leq k_1 \sum_{k=-\infty}^{\infty} |h[k]|$$

- Therefore, a DT system is stable if

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Similarly, a CT system is stable if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$