CHAP 5. Eigenvalues and eigenvectors

For a square matrix A, we set the equation

$$A x = \lambda x$$

λ: eigenvalue, x: eigenvector $(\neq 0)$

* When an eigenvector x is mapped to y by A, its direction is invariant.

* Solving the eigenvalue problem

$$(A - \lambda I) x = 0.$$

1. Compute $\det(A - \lambda I)$, which is a polynomial of degree n in λ.
2. Find the roots of the polynomial. The n roots are the eigenvalues.
3. For each eigenvalue λ, solve the equation $(A - \lambda I) x = 0$. Those solutions are the eigenvectors

Ex)

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= \lambda^2 - \lambda - 12 + 10 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) \end{aligned}$$

① $\lambda_1 = 2$

$$(A - 2I)x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} x = 0$$

$$x_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

② $\lambda_2 = -1$

$$(A + I)x = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

A^k ?

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 14 & -15 \\ 6 & -7 \end{bmatrix}$$

A^{100} ?

Note that

$$A\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$A^{100} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^{100} & \\ & (-1)^{100} \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

Ex 2) Diagonal matrix

$$A = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix} \qquad \lambda_1 = 3,\ x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda_2 = 2,\ x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex 3) Projection matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det(P - \lambda I) = \det\begin{bmatrix} \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}-\lambda \end{bmatrix} = \lambda^2 - \lambda = \lambda(\lambda - 1)$$

$\lambda_1 = 0,\ x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the points on the perpendicular line is mapped to 0.

$\lambda_2 = 1,\ x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ the points on $y=x$ is not changed.

Projection onto the line $y=x$.

Ex 4) Another projection matrix

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\lambda_0 = 1$ $\quad x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

Both x_0 and $x_1 \in R(P)$

$\lambda_1 = 0$ $\quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Both x_2 and $x_3 \in N(P)$.

Ex 5) Triangular matrix

$$A = \begin{bmatrix} 1 & 4 & 5 \\ & \frac{3}{4} & 6 \\ & & \frac{1}{2} \end{bmatrix}$$

$\det(A - \lambda I) = (1 - \lambda)(\frac{3}{4} - \lambda)(\frac{1}{2} - \lambda).$

$\therefore \lambda = 1, \frac{3}{4}, \frac{1}{2}$ // diagonal elements

Properties of eigenvalues.

① $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn} = \text{trace}(A)$

$$f(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a_{11}-\lambda & a_{12} & & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ & & \ddots & \\ a_{n1} & a_{n2} & & a_{nn}-\lambda \end{bmatrix}$$

$$f(\lambda) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + \underbrace{g(\lambda)}_{\text{order less than } n-1}$$

$$= (-\lambda)^n + (a_{11}+a_{22}+\cdots+a_{nn})(-\lambda)^{n-1} + \underbrace{h(\lambda)}_{\text{order less than } n-2} \quad \cdots ①$$

Also

$$f(\lambda) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

$$= (-\lambda)^n + (\lambda_1+\lambda_2+\cdots+\lambda_n)(-\lambda)^{n-1} + \underbrace{k(\lambda)}_{\text{order less than } 2} \quad \cdots ②$$

Since ① = ②

$$a_{11} + \cdots + a_{nn} = \lambda_1 + \cdots + \lambda_n$$

② $\lambda_1 \lambda_2 \cdots \lambda_n = \det A$

$$\det(A-\lambda I) = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

$$\Rightarrow \det A = \lambda_1 \lambda_2 \cdots \lambda_n$$

The diagonal form of a matrix

Assume that an $n \times n$ matrix A has n linearly independent eigenvectors.

$$A x_i = \lambda_i x_i \quad 1 \le i \le n \quad \text{and } x_i\text{'s are independent.}$$

$$A \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

\# $AS = S\Lambda$

or $A = S\Lambda S^{-1}$

Theorem. A matrix with distinct eigenvalues can be diagonalized, i.e., $x_1, \cdots, x_n$ are independent.

Proof) Assume that

$$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = 0$$

Then

$$\lambda_1 c_1 x_1 + \lambda_2 c_2 x_2 + \cdots + \lambda_n c_n x_n = 0$$

$$\lambda_1^2 c_1 x_1 + \lambda_2^2 c_2 x_2 + \cdots + \lambda_n^2 c_n x_n = 0$$

$$\vdots$$

$$\begin{bmatrix} c_1 x_1 & c_2 x_2 & \cdots & c_n x_n \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & & & \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{bmatrix}}_{\text{Vandermonde's matrix}} = 0$$

It is invertible if $\lambda_i \neq \lambda_j$ when $i \neq j$

By definition x_i's are nonzero vectors.

Thus, $c_1 = c_2 = \cdots = c_n = 0$

and x_i's are independent.

* Suppose that Vandermonde's matrix is not invertible

Then $\exists \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq 0$ such that $\begin{bmatrix} 1 & \lambda_1 & & \lambda_1^{n-1} \\ & & \vdots & \\ 1 & \lambda_n & & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$

This implies that

$f(x) = c_1 + c_2 x + \cdots + c_n x^{n-1}$ has n distinct solutions, which is a contradiction.

* Not all matrices are diagonalizable.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\lambda = 0 \qquad x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\lambda = 0$ is a double eigenvalue, but we have only one eigenvector.

$$B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

$\lambda = 3$ but we have only one eigenvector.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Ex 1)

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Ex 2)

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

No real eigenvalues and eigenvectors.

$\because$ The direction of a nonzero vector is changed by the rotation matrix.

$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

① $\lambda_1 = i$ $\quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

② $\lambda_2 = -i$ $\quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$

$$x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\therefore \quad A \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} i & \\ & -i \end{bmatrix}$$

Thm. If A has eigenvalues $\lambda_1, \cdots, \lambda_n$, the eigenvalues of A^2 are exactly $\lambda_1^2, \cdots, \lambda_n^2$. Also, if x is an eigenvector of A, it is also an eigenvector of A^2.

Proof)

$$A x_i = \lambda_i x_i$$

$$\Rightarrow \quad A^2 x_i = \lambda_i^2 x_i$$

Thm. If $A = S \Lambda S^{-1}$,

$$A^k = S \Lambda^k S^{-1}.$$

Thm. If A is invertible, λ_i's are nonzero, and A^{-1} has eigenvalues $\frac{1}{\lambda_i}$.

$$A x_i = \lambda_i x_i \quad \Rightarrow \quad \frac{1}{\lambda_i} x_i = A^{-1} x_i$$

Ex)

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \lambda_1 = i \qquad x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\lambda_2 = -i \qquad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$K^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\lambda_1 = -1$ (double root)

$$x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\lambda_1 = \frac{1}{i} = -i \qquad x_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\lambda_2 = \frac{1}{-i} = i \qquad x_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Difference Equations

① Fibonacci sequence

$0, 1, 1, 2, 3, 5, 8, \cdots$

$F_{k+2} = F_{k+1} + F_k, \quad F_0 = 0, \quad F_1 = 1.$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

$$= A u_k$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$u_1 = A u_0$

$u_2 = A u_1 = A^2 u_0$

$\vdots$

$u_k = A^k u_0$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \qquad \lambda_1 \lambda_2 = -1$$

$$\lambda_1 \triangleq \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 1 \\ 1 & \frac{-1+\sqrt{5}}{2} \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\therefore \quad A \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$$

$$A = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \\ & \lambda_2^k \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\therefore\ u_k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \\ & \lambda_2^k \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & \\ & \lambda_2^k \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k \\ -\lambda_2^k \end{bmatrix}$$

$$\therefore\ F_k = \frac{1}{\sqrt{5}} \left[\lambda_1^k - \lambda_2^k \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right]$$

$$\text{Golden ratio} = \lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \frac{1+\sqrt{5}}{2}$$

$$\doteq 1.618.$$

In general, (if A is diagonalizable?)

$$\begin{cases} u_k = A^k u_0 \\ A^k = S \Lambda^k S^{-1} \end{cases}$$

Therefore,

$$u_k = S \Lambda^k S^{-1} u_0$$

let

$$S^{-1} u_0 \triangleq \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c \quad \cdots (b)$$

Then

$$u_k = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k c_1 \\ \vdots \\ \lambda_n^k c_n \end{bmatrix}$$

$$= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \cdots + c_n \lambda_n^k x_n \quad \cdots (a)$$

Another interpretation of (a)

From (b),

$$u_0 = S c$$

$$= c_1 x_1 + \cdots + c_n x_n.$$

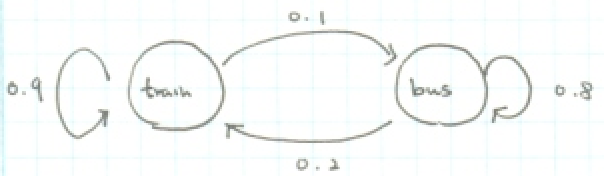
Then

$$A^k u_0 = c_1 A^k x_1 + \cdots + c_n A^k x_n$$

$$= c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n$$

stability.

② Markov Process



y_k # of train users on day k

z_k # of bus users on day k.

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = \begin{bmatrix} \underset{\Sigma = 1}{\boxed{\begin{matrix} 0.9 \\ 0.1 \end{matrix}}} & \underset{\Sigma = 1}{\boxed{\begin{matrix} 0.2 \\ 0.8 \end{matrix}}} \end{bmatrix} \begin{bmatrix} y_{k-1} \\ z_{k-1} \end{bmatrix}$$

All entries are nonnegative.

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

where $A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$

Diagonalization of A

Characteristic polynomial

$$= \det(A - \lambda I)$$

$$= (0.9-\lambda)(0.8-\lambda) - 0.02$$

$$= \lambda^2 - 1.7\lambda + 0.7$$

$$= (\lambda - 1)(\lambda - 0.7).$$

$\lambda_1 = 1$ $\quad A - \lambda_1 I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}$

$\therefore \quad x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = 0.7$ $\quad A - \lambda_2 I = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$

$$A \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \\ & 0.7 \end{bmatrix}$$

$$A = S \begin{bmatrix} 1 & \\ & 0.7 \end{bmatrix} S^{-1}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = S \begin{bmatrix} 1 & \\ & (0.7)^k \end{bmatrix} S^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$= S \begin{bmatrix} 1 & \\ & (0.7)^k \end{bmatrix} \times -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

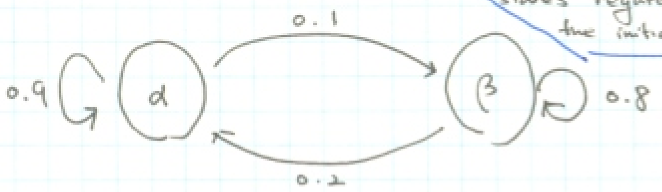
$$= \frac{1}{3} S \begin{bmatrix} 1 & \\ & (0.7)^k \end{bmatrix} \begin{bmatrix} y_0 + z_0 \\ y_0 - 2z_0 \end{bmatrix}$$

$$= \frac{1}{3} S \begin{bmatrix} y_0 + z_0 \\ (0.7)^k (y_0 - 2z_0) \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} (y_0 + z_0) + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} (0.7)^k (y_0 - 2z_0)$$

$$\Rightarrow \begin{bmatrix} \frac{2(y_0 + z_0)}{3} \\ \frac{(y_0 + z_0)}{3} \end{bmatrix} \quad \text{as } k \to \infty .$$

It goes to the steady states regardless of the initial conditions



$$\alpha + \beta = y_0 + z_0$$

$$\alpha \cdot 0.1 = \beta \cdot 0.2 .$$

$$\therefore \alpha = \frac{2(y_0 + z_0)}{3} \qquad \beta = \frac{y_0 + z_0}{3}$$

* General properties of Markov processes

(a) $\lambda_1 = 1$ is an eigenvalue

(b) Its eigenvector x_1 is nonnegative and it is a steady state since $A x_1 = x_1$

(c) The other eigenvalues satisfy $|\lambda_i| \leq 1$.

(d) If any power of A has all positive entries, these other $|\lambda_i|$ are strictly less than 1. The solution $A^k u_0$ approaches a multiple of x_1.

beyond the scope of this course.

Extension from $\mathbb{R}^n$ to $\mathbb{C}^n$

* length of $x = (1, i)$?

$x^T x = 1^2 + i^2 = 0$?

* $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$\boxed{\begin{aligned} \|x\|^2 &= |x_1|^2 + |x_2|^2 + |x_3|^2 + \cdots + |x_n|^2 \\ &= \bar{x}^T x \end{aligned}} \cdots (1)$$

where $\bar{x}$ denotes the complex conjugate of x, i.e.

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

For example.

If $x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$\begin{aligned} \|x\|^2 &= \bar{x}^T x \\ &= \begin{bmatrix} 1 \\ -i \end{bmatrix}^T \begin{bmatrix} 1 \\ i \end{bmatrix} \\ &= 1 - i^2 = 2 \end{aligned}$$

* Inner product between x and y

$$\boxed{\bar{x}^T y = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n} \cdots (2)$$

This is consistent with our definition of length

$\|x\|^2$ = the inner product of x with itself.

Also, if both x and y are real

$$\bar{x}^T y = x^T y$$

Thus, it is consistent with our definition of inner product in $\mathbb{R}^n$.

* x and y are orthogonal (perpendicular) if

$$\bar{x}^T y = 0 .$$

Ex) $x = (1+i, 3i)$ $y = (4, 2-i)$

$$\begin{aligned} \bar{x}^T y &= (1-i)\,4 - 3i(2-i) \\ &= 1 - 10i \end{aligned}$$

x and y are not orthogonal.

In both ① and ②, we see

$\bar{x}^T$, i.e., the combined operation of conjugate and transpose.

Thus, we define, for any matrix A,

$$A^H = \bar{A}^T$$

which is called "A conjugate transpose."

As a rule of thumb, the transpose in $\mathbb{R}^n$ is converted to the conjugate transpose in $\mathbb{C}^n$

ⓐ Length^2 $\quad x^T x \rightarrow x^H x$

ⓑ Innerproduct $\quad x^T y \rightarrow x^H y$

Ex)

$$A = \begin{bmatrix} 2+i & 3i \\ 4-i & 5 \\ 0 & 0 \end{bmatrix}$$

$$A^H = \begin{bmatrix} 2-i & 4+i & 0 \\ -3i & 5 & 0 \end{bmatrix}$$

Also, note that $\boxed{(AB)^H = B^H A^H}$

$$\because \quad (AB)^H = (\overline{AB})^T = (\bar{A}\,\bar{B})^T$$

$$= \bar{B}^T \bar{A}^T = B^H A^H.$$

Hermitian Matrices

$$A^H = A$$

Ex) $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$

Properties:

① Diagonal elements are real.

② A real symmetric matrix is Hermitian.

$[\because\ A^H = \bar{A}^T = A^T = A.]$

③ For all complex vectors x,

$x^H A x$ is real.

ex) $[\bar{u}\ \ \bar{v}]\begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

$= \underbrace{2\bar{u}u}_{\text{real}} + \underbrace{\bar{u}(3-3i)v + \bar{v}(3+3i)u}_{\text{complex conjugate}} + \underbrace{5v\bar{v}}_{\text{real}}$

In general,

$y = x^H A x = y^H = x^H A^H x$

$\therefore\ y = \bar{y}$ $\quad y$ is real.

④ Every eigenvalue is real.

$\because$ Suppose $Ax = \lambda x$

$x^H A x = x^H \lambda x = \lambda \|x\|^2$

$\lambda = \dfrac{x^H A x}{\|x\|^2}$ is real.

ex) $\det\begin{bmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{bmatrix} = \lambda^2 - 7\lambda + 10 - 18$

$= (\lambda-8)(\lambda+1)$

$\lambda = 8$ or -1

⑤ A has an orthonormal set of eigenvectors.

Proof) For simplicity, assume that all eigenvalues are distinct.

$Ax_1 = \lambda_1 x_1, \quad Ax_2 = \lambda_2 x_2$

$x_2^H A x_1 = \lambda_1 x_2^H x_1$

$x_1^H A x_2 = \lambda_2 x_1^H x_2$

$x_2^H A x_1 = \lambda_2 x_2^H x_1$

$\therefore\ (\lambda_1 - \lambda_2) x_2^H x_1 = 0$

$\therefore\ x_2^H x_1 = 0.$

They are orthogonal, and we can normalize them by

$\dfrac{x_1}{\|x_1\|}, \dfrac{x_2}{\|x_2\|}, \ldots$

ex) $A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$

$\lambda_1 = 8$

$A - \lambda_1 I = \begin{bmatrix} -6 & 3-3i \\ 3+3i & -3 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ or $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$\lambda_2 = -1$

$A - \lambda_2 I = \begin{bmatrix} 3 & 3-3i \\ 3+3i & 6 \end{bmatrix}$

$x_2 = \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$ or $\frac{1}{\sqrt{3}} \begin{bmatrix} 1-i \\ -1 \end{bmatrix}$

$x_1^H x_2 = 1-i - (1-i) = 0.$

⑥ $A \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

where $x_i^H x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$A \underbrace{U}_{\text{unitary matrix}} = U \Lambda$ $U^H U = \begin{bmatrix} x_1^H \\ \vdots \\ x_n^H \end{bmatrix} \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = I.$

$A = U \Lambda U^{-1}$

$= U \Lambda U^H$

⑦ If A is real and symmetric, the eigenvectors are also real.

($\because$ They are solutions of $\underbrace{(A - \lambda I)}_{\text{real}} x = 0$)

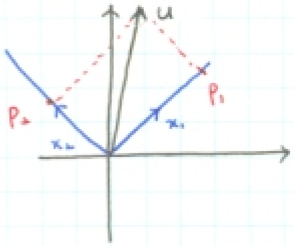
Then $A = U \Lambda U^H$

$= \underbrace{Q}_{\text{orthogonal matrix}} \Lambda Q^T \quad \cdots (i)$

$A = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$

$= \lambda_1 \underbrace{x_1 x_1^T}_{\text{projection onto } x_1\text{-axis}} + \cdots + \lambda_n x_n x_n^T$

For example,



$Au = \lambda_1 x_1 x_1^T u + \lambda_2 x_2 x_2^T u$

$= \lambda_1 P_1 + \lambda_2 P_2.$

Considering x_1 and x_2 as new coordinate axes, A scales the input u by a factor of λ_1 along x_1 axis, λ_2 along x_2 axis

(i) is called also as principal axis theorem

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

$$\lambda_1 = 1 \qquad x_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = 3 \qquad x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & \\ & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= 1 \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 3 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

To summarize, a Hermitian matrix has real eigenvalues and orthonormal set of eigenvectors.

It is diagonalizable

$$A = U \Lambda U^H \qquad U \text{ unitary matrix}$$

$$= Q \Lambda Q^H \qquad Q \text{ orthogonal matrix}$$

(when A is real and symmetric)

Unitary Matrices

$$U^H U = I.$$

Properties :

① $(Ux)^H Uy = x^H y$

(angle and length are preserved)

② Every eigenvalue has absolute value $|\lambda| = 1$

$\because$ $Ux = \lambda x$

$\|Ux\| = \|x\| = |\lambda| \|x\|$

$\Rightarrow$ $|\lambda| = 1$

③ Eigenvectors, corresponding to different eigenvalues, are orthogonal.

Let $Ux_1 = \lambda_1 x_1$, $Ux_2 = \lambda_2 x_2$

$$(Ux_1)^H (Ux_2) = x_1^H x_2$$
$$= \bar{\lambda}_1 \lambda_2 x_1^H x_2$$

$\therefore$ $(\bar{\lambda}_1 \lambda_2 - 1) x_1^H x_2 = 0$

$(e^{-i\theta_1} e^{i\theta_2} - 1) x_1^H x_2 = 0$

$\therefore$ $x_1^H x_2 = 0$ ($\because$ $\theta_2 - \theta_1 \neq 2n\pi$)

Ex)

$$U = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$$\det(U - \lambda I) = (\lambda - \cos t)^2 + \sin^2 t$$
$$= \lambda^2 - 2\cos t \, \lambda + 1$$

$$\lambda = \cos t \pm \sqrt{\cos^2 t - 1} = \cos t \pm i \sin t = e^{\pm it}$$

$\lambda_1 = e^{it}$ $\quad x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$\lambda_2 = e^{-it}$ $\quad x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$

$$\therefore \quad U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{it} & \\ & e^{-it} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$$

Note that $x_1^H x_2 = \frac{1}{2}(1-1) = 0.$

$|\lambda_1| = |\lambda_2| = 1.$