Chapter 13. Complex Numbers and Functions

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The contents herein are based on the book "Advanced Engineering Mathematics" by E. Kreyszig and only for the course KEEE202, Korea University.

I. Complex Numbers

We introduce the imaginary unit i, which is defined by

$$i^2 = -1.$$

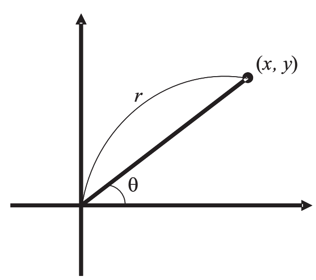
Let $z = x + iy$ denote a complex number, where x and y are real numbers. Then its conjugate is defined by

$$\bar{z} = z^* = x - iy.$$

We can easily see that

- Re $z = x = \frac{z+\bar{z}}{2}$
- Im $z = y = \frac{z-\bar{z}}{2i}$
- z is real $\Leftrightarrow z = \bar{z}$
- z is purely imaginary $\Leftrightarrow z = -\bar{z}$

⋆ Geometric interpretation:



Note that $z = x + iy$ can be thought as a point (x, y) in the Cartesian coordinate. The same number can be seen as a point (r, θ) also in the polar coordinate, where

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta.$$

Thus, we have

$$\begin{aligned} z &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \end{aligned}$$

where we use the Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

- Euler's formula is natural in the sense that it satisfies

$$e^{x+y} = e^x e^y.$$

- For a general $z = x + iy$, we define

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= e^x(\cos y + i \sin y). \end{aligned}$$

- Easy multiplication in polar form: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}.$$

- What is $\sqrt[4]{1}$?
  - In general, $\sqrt[n]{1} = e^{i\frac{2\pi}{n}k}$, $(0 \le k \le n-1)$.

II. Complex Functions

A complex function is given by

$$w = f(z).$$

Let $z = x + iy$ and $w = u + iv$. Then, we have

$$w = f(z) = u(x, y) + iv(x, y).$$

⋆ Example:

$$\begin{aligned} w = f(z) &= z^2 \\ &= (x + iy)^2 \\ &= x^2 - y^2 + 2ixy. \end{aligned}$$

Therefore

$$\begin{aligned} u(x, y) &= x^2 - y^2, \\ v(x, y) &= 2xy. \end{aligned}$$

• Limit

$$\lim_{z \to z_0} f(z) = l.$$

For every $\epsilon > 0$, we have a $\delta > 0$ such that, if $|z - z_0| < \delta$ and $z \neq z_0$, then $|f(z) - l| < \epsilon$. Intuitively speaking, as z approaches z_0 from any direction, $f(z)$ gets closer to l.

⋆ Example: Show that $\lim_{z \to 0} z^2 = 0$.

• Continuity:

A function $f(z)$ is said to be continuous at $z = z_0$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

• Derivative:

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \end{aligned}$$

If the limit exists, f is differentiable at $z = z_0$.

⋆ Example 1: Find the derivative of $f(z) = z^2$.

⋆ Example 2: Show that $f(z) = \bar{z}$ is not differentiable.

• Analyticity:

$f(z)$ is said to be analytic in a domain D if it is defined and differentiable at all points in D. $f(z)$ is said to be analytic at a point $z = z_0$ if it is analytic in a neighborhood of z_0.

⋆ Terminology:

• Neighborhood of a: an open disk around a, i.e., $\{z : |z - a| < \rho\}$.

• Open: A set S is called open if every point of S has a neighborhood consisting of points that belong to S only.

• Connectedness: A set S is called connected if any two of its points can be connected by a curve all of whose points belong to S.

• Domain: an open and connected set.

• Cauchy-Riemann equation:

A function $f(z) = u(x, y) + iv(x, y)$ is analytic if and only if

$$u_x = v_y \quad \text{and} \quad u_y = -v_x.$$

Also, the derivative is given by

$$\begin{aligned} f'(z) &= u_x(x, y) + iv_x(x, y) \\ &= v_y(x, y) - iu_y(x, y). \end{aligned}$$

⋆ Example 1: $f(z) = z^2$.

⋆ Example 2: $f(z) = e^z$.

Proof)

• Laplace equation:

If $f(z) = u(x, y) + iv(x, y)$ is analytic, both u and v satisfy Laplace's equation. In other words,

$$\begin{aligned} \nabla^2 u &= u_{xx} + u_{yy} = 0, \\ \nabla^2 v &= v_{xx} + v_{yy} = 0. \end{aligned}$$

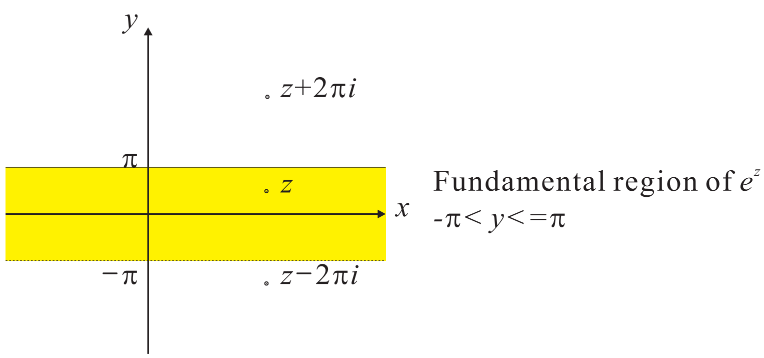
Proof)

III. Exponential Function

$$f(z) = e^z = e^x(\cos y + i \sin y)$$

Properties)

- $e^z = e^x$ for $z = x + i0$.
- $e^{iy} = \cos y + i \sin y$ (Euler's formula)
- $|e^z| = e^x$.
- $e^z \neq 0$.
- e^z is analytic for all z, i.e., it is an entire function.
- $(e^z)' = e^z$.
- $e^{z_1} e^{z_2} = e^{z_1+z_2}$.
- $e^{z+2\pi i} = e^z$.



⋆ Example: $e^z = -2$. What is z?

IV. Trigonometric and Hyperbolic Functions

• Note that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. We extend these relationships to general complex numbers by

$$\begin{aligned}
\cos z &= \frac{e^{iz}+e^{-iz}}{2}, \\
\sin z &= \frac{e^{iz}-e^{-iz}}{2i}, \\
\tan z &= \frac{\sin z}{\cos z}.
\end{aligned}$$

All properties we know about real trigonometric functions extend in a straightforward manner to the complex counterparts.

⋆ Example:

$$(\sin z)' = \frac{ie^{iz}+ie^{-iz}}{2i} = \cos z.$$

⋆ Computing $\cos z$:

$$\begin{aligned}
\cos z &= \frac{e^{iz}+e^{-iz}}{2}, \\
&= \frac{1}{2}\left[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right] \\
&= \frac{1}{2}(e^{-y}+e^{y})\cos x - i\frac{1}{2}(e^{y}-e^{-y})\sin x \\
&= \cosh y \cos x - i \sinh y \sin x.
\end{aligned}$$

• Hyper cosine and sine are defined by

$$\begin{aligned}
\cosh z &= \frac{e^{z}+e^{-z}}{2}, \\
\sinh z &= \frac{e^{z}-e^{-z}}{2}.
\end{aligned}$$

• We have the relationships between the trignometric and the hyperbolic functions.

$$\begin{aligned}
\cosh iz &= \cos z, \\
\sinh iz &= i\sin z.
\end{aligned}$$

V. Logarithm

$$\mathrm{Ln}\,z = \ln|z| + i\mathrm{Arg}\,z, \qquad (-\pi < \mathrm{Arg}\,z \leq \pi). \tag{1}$$

⋆ Derivation of logarithmic function:

Note that the logarithm is the inverse of the exponential function. Thus,

$$w = \ln z \quad \Rightarrow \quad z = e^w.$$

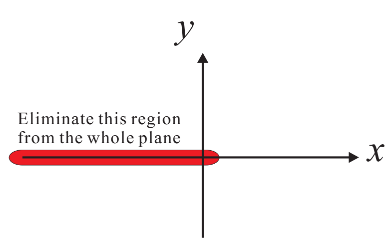
Let $z = re^{i\theta}$ and $w = u + iv$. Then, $re^{i\theta} = e^{u+iv} = e^u e^{iv}$. Therefore, we have $e^u = r$ and $v = \theta + 2n\pi$, where n is an integer. Therefore,

$$\begin{aligned} w &= \ln z \\ &= u + iv \\ &= \ln r + i(\theta + 2n\pi) \\ &= \ln r + i(\arg z + 2n\pi) \end{aligned}$$

The imaginary part v is not uniquely defined. If we constrain it to be a principal value between $-\pi$ and π, we come to the definition in (1).

Properties:

1. For negative real z, $\mathrm{Ln}\,z = \ln|z| + i\pi$.
2. $e^{\mathrm{Ln}\,z} = z$.
3. $(\mathrm{Ln}\,z)' = \frac{1}{z}$.



⋆ Example: Evaluate $\mathrm{Ln}\,(3 - 4i)$.

VI. General Powers

$$z^c = e^{\operatorname{Ln} z^c} = e^{c \operatorname{Ln} z}$$

⋆ Example: Evaluate i^i.