Chapter 16. Laurent Series

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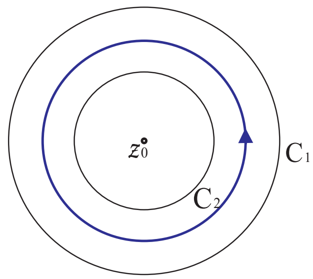
The contents herein are based on the book "Advanced Engineering Mathematics" by E. Kreyszig and only for the course KEEE202, Korea University.

I. Laurent's Theorem

Laurent series generalize Taylor series.

⋆ Laurent Theorem: Let $f(z)$ be analytic in a domain, which contains C_1 and C_2 and the annulus between them. Then, we have



$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*. \quad (1)$$

• The series converges and represents $f(z)$ in the enlarged open annulus, obtained by continuously increasing C_1 and decreasing C_2, until they reach a point where $f(z)$ is singular.

• If z_0 is the only singular point inside C_2, C_2 can be shrunk to the point z_0. In other words, $f(z)$ converges in a disk except z_0. Also, in such a case, the negative powers are called the principal part.

⋆ Ex 1 : $z^{-5} \sin z$

⋆ Ex 2 : $z^2 e^{\frac{1}{z}}$

⋆ Ex 3 : $\frac{1}{1-z}$

⋆ Ex 4 : Find all Laurent series of

$$f(z) = \frac{-2z+3}{z^2-3z+2} \qquad \text{with center } 0$$

II. Singularities and Zeros

• $f(z)$ is said to be *singular* at $z = z_0$, if $f(z)$ is not analytic at $z = z_0$ but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. Also, a singular point z_0 is called *isolated*, if z_0 has a neighborhood without further singularities.

⋆ Ex: $\tan \frac{1}{z}$ is singular at $z = 0$. But, it is not an isolated singularity.

⋆ Classification of isolated singularities at $z = z_0$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}_{\text{principal part}} \qquad (0 < |z - z_0| < R).$$

If the principal part contains finitely many terms, $z = z_0$ is called a pole and we have

$$\text{Principal part } = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \ldots + \frac{b_m}{(z-z_0)^m}$$

$$m = \text{order of the pole at } z = z_0$$

Especially, a pole of the first order is called a simple pole. If the principal part contains infinitely many terms, $z = z_0$ is called an isolated essential singularity.

⋆ Ex 1:

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

$z = 0$: a simple pole.

$z = 2$: a pole of order 5.

⋆ Ex 2:

$$f(z) = \sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!} \frac{1}{z^3} + \frac{1}{5!} \frac{1}{z^5} + \ldots$$

$z = 0$: an isolated essential singularity.

• A zero of an analytic function $f(z)$ in D is a point $z = z_0$ such that $f(z_0) = 0$. A zero has order n, if $f, f', f'', \ldots, f^{(n-1)}$ are zero at $z = z_0$ but $f^{(n)}(z_0) \neq 0$. A first-order zero is called a simple zero.

⋆ Ex)

- $f(z) = 1 + z^2$

- $f(z) = (1 - z^4)^2$

- $f(z) = (1 - \cos z)^2$

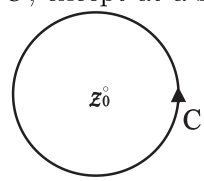
• Taylor series at a zero of order n is given by

$$\begin{aligned} f(z) &= a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \ldots \qquad (a_n \neq 0) \\ &= (z - z_0)^n \left[a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \ldots\right] \end{aligned}$$

• Relationship between poles and zeros: If $f(z)$ has a zero of order n at $z = z_0$, $\frac{1}{f(z)}$ has a pole of order n at $z = z_0$.

III. Residue Integration Method

⋆ Let $f(z)$ be analytic on C and inside C, except at a singular point $z = z_0$.



Then, we have

$$\int_C f(z)dz = 2\pi i b_1$$

where b_1 is called the residue of $f(z)$ at $z = z_0$ and denoted by

$$b_1 = \text{Res}_{z=z_0} f(z).$$

Proof)

⋆ Ex 1: $f(z) = z^{-4} \sin z$, C: counterclockwise unit circle.

⋆ Ex 2: $f(z) = \frac{1}{z^3 - z^4}$, C: $z = \frac{1}{2}$, clockwise.

IV. Formulas for Residues

⋆ Simple poles:

$$\mathrm{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to\infty} (z - z_0) f(z).$$

Also, if $f(z) = \frac{p(z)}{q(z)}$ and $p(z_0) \neq 0$, then $q(z)$ has a simple zero at z_0, and

$$\mathrm{Res}_{z=z_0} f(z) = \mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

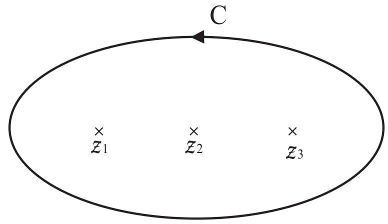
⋆ Ex: $f(z) = \frac{az+i}{z^3+z}$

⋆ Poles of order m:

$$\mathrm{Res}_{z\to z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \Big[(z - z_0)^m f(z) \Big]$$

⋆ Ex: $f(z) = \frac{50z}{(z-1)^2(z+4)}$

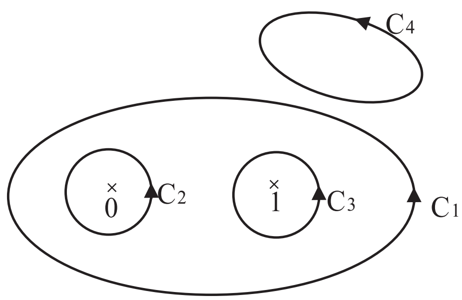
V. Multiple Singularities Inside Contour



$$\int_C f(z)dz = 2\pi i\Big[\mathrm{Res}_{z=z_1} f(z) + \mathrm{Res}_{z=z_2} f(z) + \mathrm{Res}_{z=z_3} f(z)\Big]$$

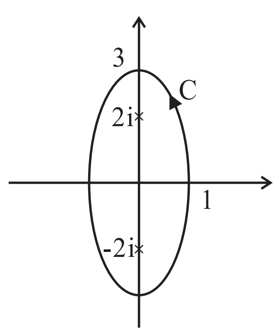
⋆ Ex 1:

$$f(z) = \int_C \frac{4-3z}{z^2-z}dz$$



⋆ Ex 2 :

$$f(z) = \frac{ze^{\pi z}}{z^4-16} + ze^{\frac{\pi}{z}}$$



VI. Residue Integration of Real Integrals

A. *Type 1 - Integrals Including Sinusoidal Functions*

To evaluate

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta,$$

we set $z = e^{i\theta}$. Then, we have

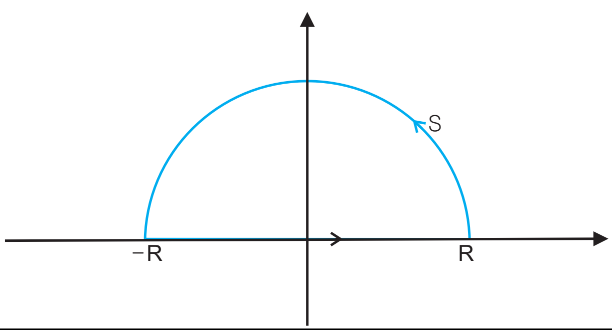
$$J = F\left(\frac{1}{2}\Big(z + \frac{1}{z}\Big), \quad \frac{1}{2i}\Big(z - \frac{1}{z}\Big)\right)\frac{1}{iz}dz$$

⋆ Ex) $\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta} d\theta$

B. *Type 2 - Real Integral over The Whole Line*

Suppose that $f(x)$ is a real rational function, whose denominator $\neq 0$ for all x and has a degree at least two units higher than the degree of denominator. Then, we have

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{\text{u.h.p.}} \text{Res} f(z).$$



⋆ Ex) $\int_0^{\infty} \frac{1}{1+x^4} dx$

C. Type 3 - Fourier Integrals

Suppose that $f(x)$ is a real rational function, whose denominator $\neq 0$ for all x and has a degree at least two units higher than the degree of denominator. Then, for $s > 0$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\cos sx\,dx &= -2\pi \sum_{\text{u.h.p.}} \operatorname{Im}\operatorname{Res}\left[f(z)e^{isz}\right],\\ \int_{-\infty}^{\infty} f(x)\sin sx\,dx &= 2\pi \sum_{\text{u.h.p.}} \operatorname{Re}\operatorname{Res}\left[f(z)e^{isz}\right].\end{aligned}$$

Alternatively,

$$\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = 2\pi i \sum_{\text{u.h.p.}} \operatorname{Res}\left[f(z)e^{isz}\right].$$

⋆ Ex: evaluate $\int_{-\infty}^{\infty} \frac{\cos sx}{k^2+x^2}dx$ and $\int_{-\infty}^{\infty} \frac{\sin sx}{k^2+x^2}dx$, where $s > 0$ and $k > 0$.

D. Type 4 - Cauchy Principal Value

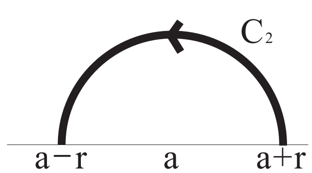
Suppose that $A < a < B$ and $\lim_{x\to a} |f(x)| = \infty$. Then, the Cuchy principal value is defined by

$$\text{pr.v.} \int_A^B f(x)dx = \lim_{\epsilon\to 0}\left[\int_A^{a-\epsilon} f(x)dx + \int_{a+\epsilon}^{B} f(x)dx\right].$$

The following theorem is useful in computing the Cauchy principal value.

⋆ If $f(z)$ has a simple pole at $z = a$ on the real axis,

$$\lim_{r\to 0}\int_{C_2} f(z)dz = \pi i \operatorname{Res}_{z=a} f(z).$$



⋆ Ex: Evaluate

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{1}{(x^2 - 3x + 2)(x^2 + 1)} dx$$