Digital Image Processing

Sampling

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Sampling and Quantization

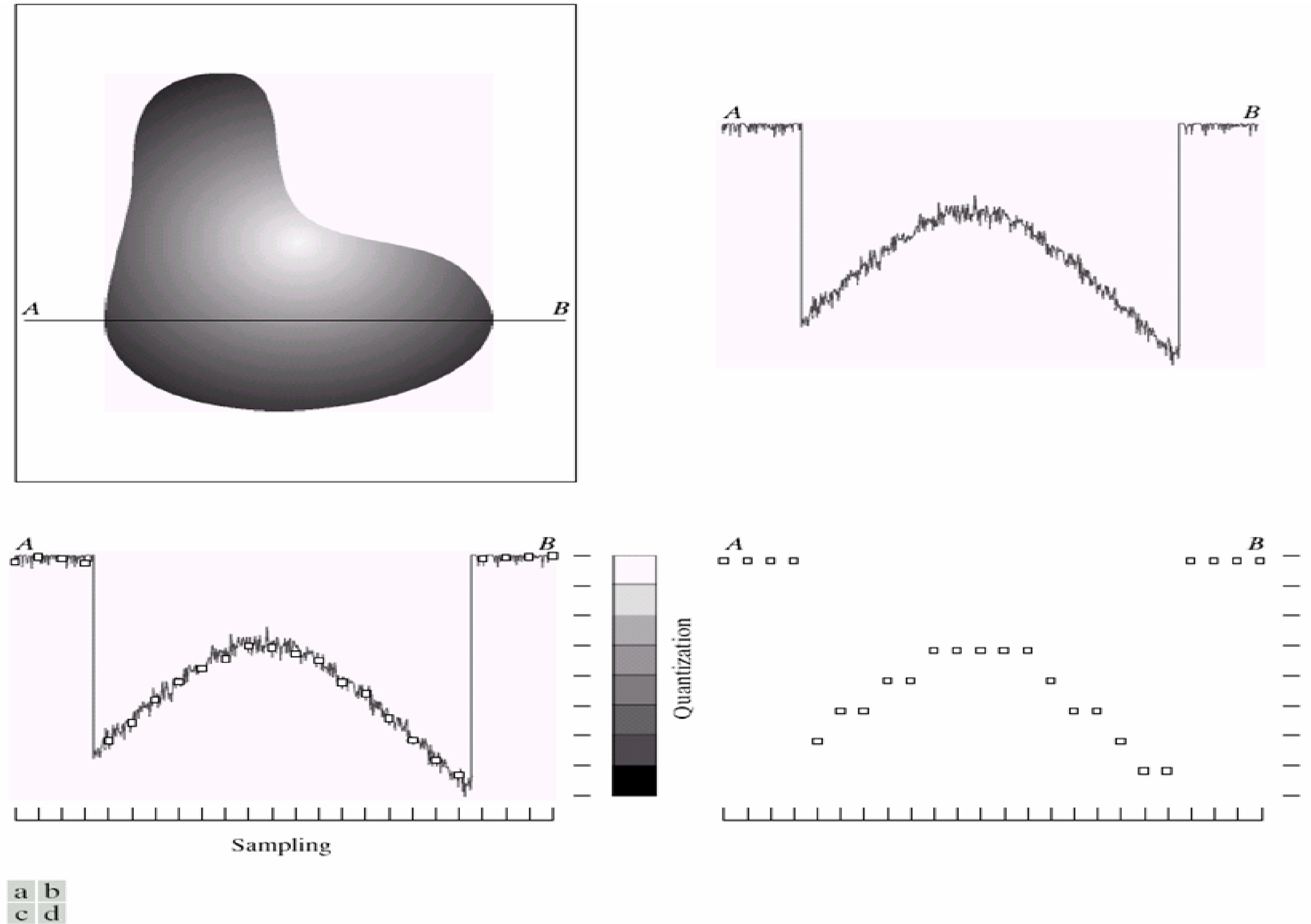


FIGURE 2.16 Generating a digital image. (a) Continuous image. (b) A scan line from A to B in the continuous image, used to illustrate the concepts of sampling and quantization. (c) Sampling and quantization. (d) Digital scan line.

2-D Function

- Continuous function (analog image)

$$f(x, y) : x, y \in \mathbb{R}$$

- Discrete function (sampled image)

$$u(m, n) : m, n \in \mathbb{Z}$$

Delta Function in 1-D Discrete Case

- Delta function

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

- Arbitrary function as sum of shifted delta functions

$$u(n) = \sum_{\bar{n}=-\infty}^{\infty} u(\bar{n})\delta(n - \bar{n})$$

Delta Function in 2-D Discrete Case

- Delta function

$$\delta(m, n) = \delta(m) \cdot \delta(n)$$

- Arbitrary function as sum of shifted delta functions

$$u(m, n) = \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m}, \bar{n}) \delta(m - \bar{m}, n - \bar{n})$$

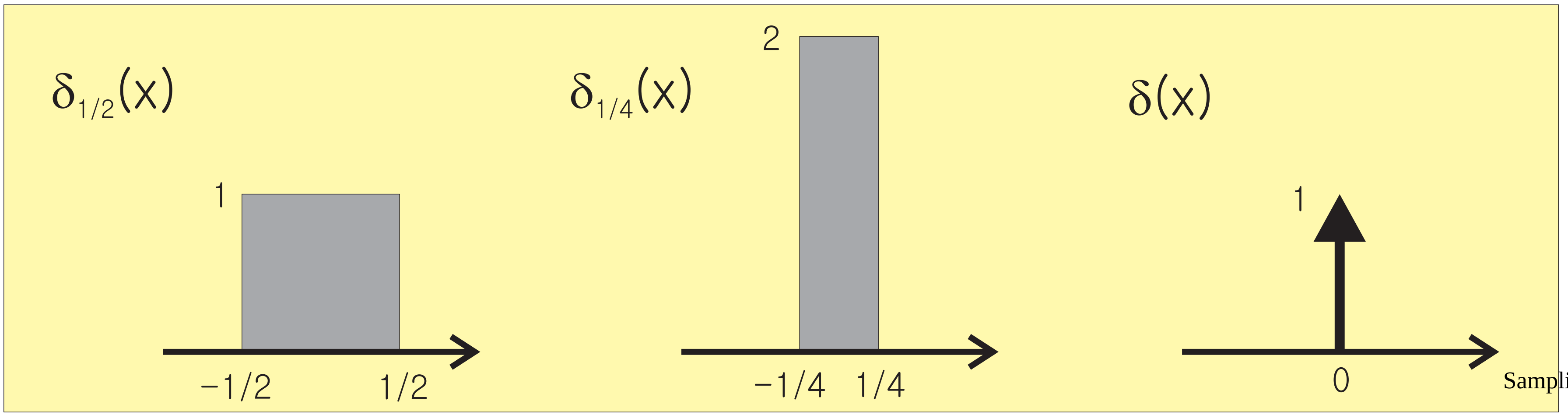
Delta Function in 1-D Continuous Case

- Approximate delta function

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{2\epsilon} & \text{if } |x| < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

- Delta function

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon(x)$$



Delta Function in 1-D Continuous Case

- Arbitrary function as sum (integral) of shifted delta functions

$$f(x) = \int_{-\infty}^{\infty} f(\bar{x})\delta(x - \bar{x})d\bar{x}$$

- Proof

$$\begin{aligned}
\int_{-\infty}^{\infty} f(\bar{x})\delta(x - \bar{x})d\bar{x} &= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(\bar{x})\delta_{\epsilon}(x - \bar{x})d\bar{x} \\
&= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{x-\epsilon}^{x+\epsilon} f(\bar{x})d\bar{x} \\
&= f(x)
\end{aligned}$$

Delta Function in 2-D Continuous Case

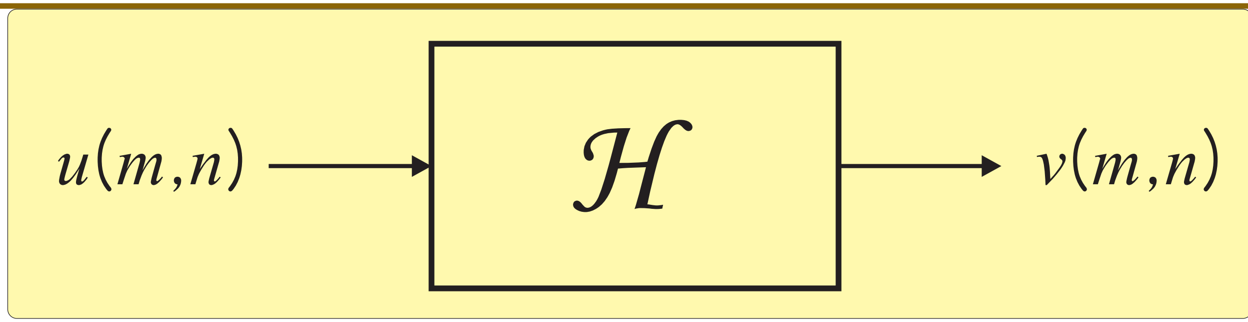
- Delta function

$$\delta(x, y) = \delta(x) \cdot \delta(y)$$

- Arbitrary function as sum of shifted delta functions

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\bar{x}, \bar{y}) \delta(x - \bar{x}, y - \bar{y}) d\bar{x} d\bar{y}$$

Linear Shift Invariant System



- Linearity

$$\begin{aligned} \mathcal{H}[u_1(m,n)] &= v_1(m,n), \\ \mathcal{H}[u_2(m,n)] &= v_2(m,n) \\ \Rightarrow \mathcal{H}[a_1 u_1(m,n) + a_2 u_2(m,n)] \\ &= a_1 v_1(m,n) + a_2 v_2(m,n) \end{aligned}$$

- Shift Invariance

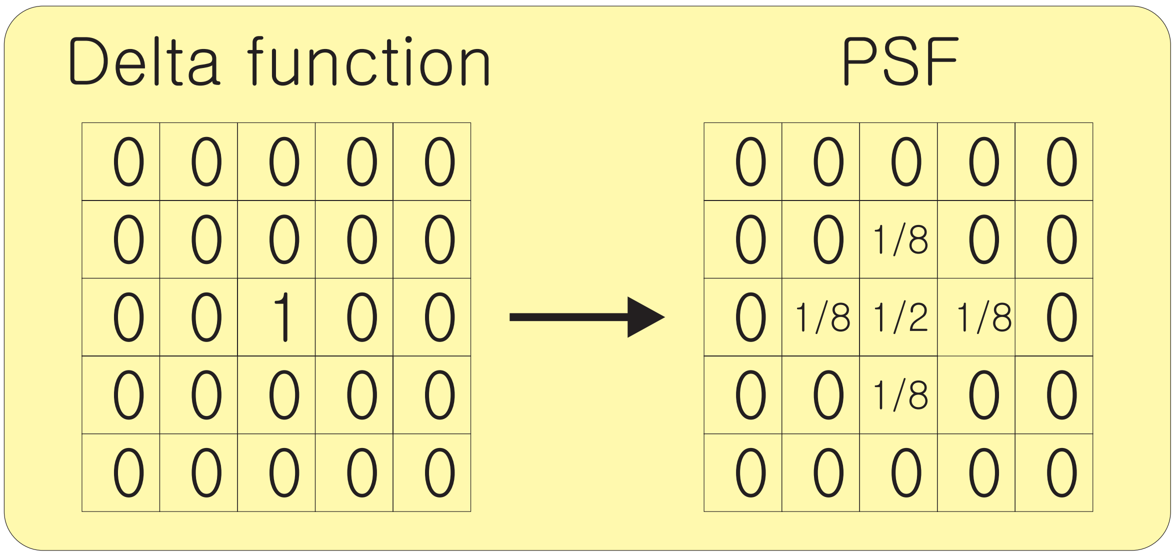
$$\begin{aligned} \mathcal{H}[u(m,n)] &= v(m,n) \\ \Rightarrow \mathcal{H}[u(m-\bar{m}, n-\bar{n})] &= v(m-\bar{m}, n-\bar{n}) \end{aligned}$$

Impulse Response

- Consider an LSI system $\mathcal{H}$ such that

$$\mathcal{H}(\delta(m,n)) = h(m,n)$$

- $h(m,n)$ is called
  - impulse response
  - point spread function (PSF)



Convolution

- Given an LSI system $\mathcal{H}$ with PSF $h(m,n)$ and an arbitrary input $u(m,n)$, what is the output $v(m,n)$?
- Recall that

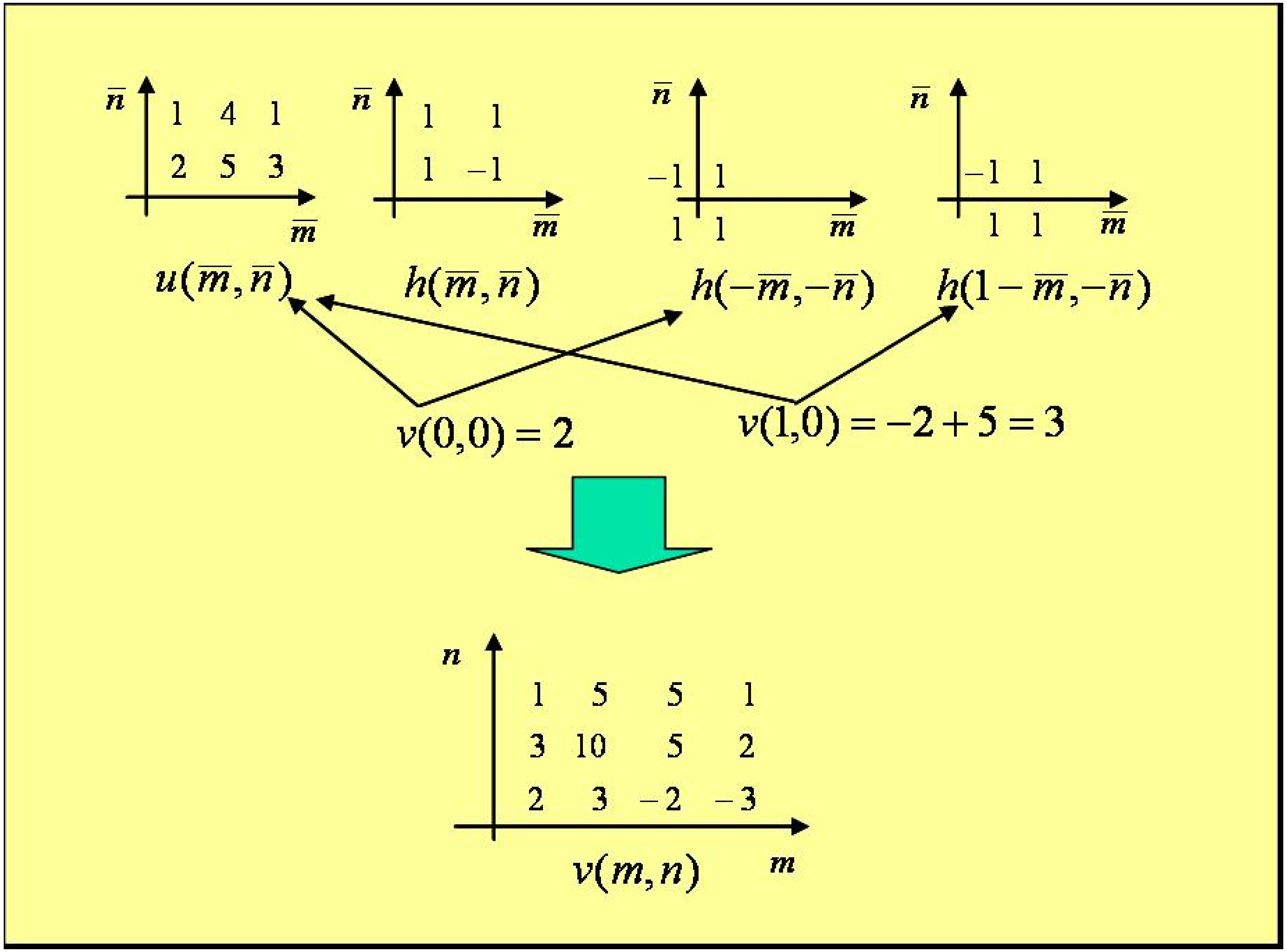
$$u(m,n) = \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m},\bar{n})\delta(m-\bar{m}, n-\bar{n})$$

- Hence

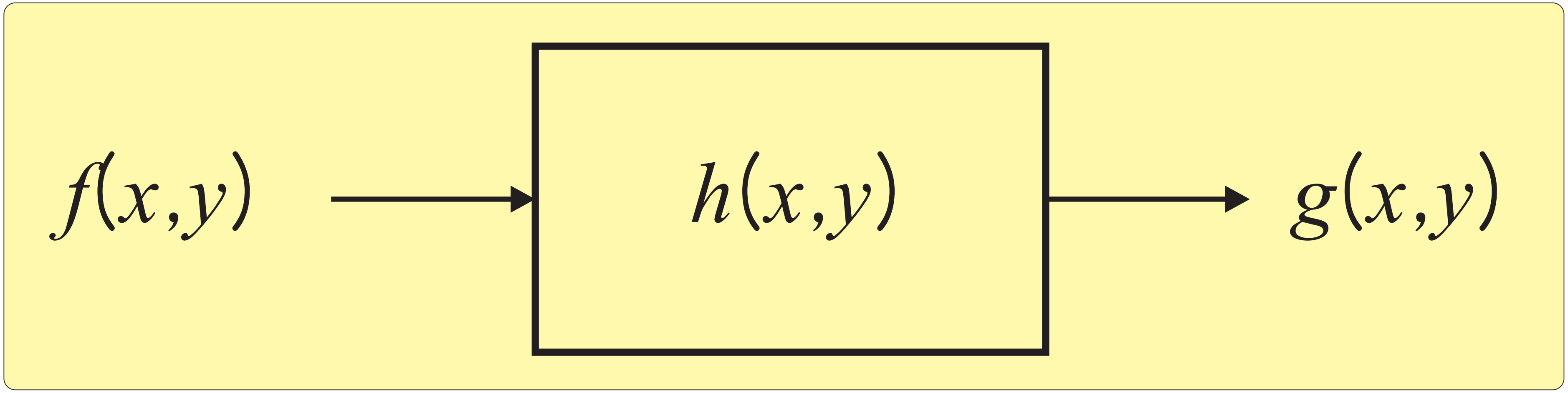
$$\begin{aligned} v(m,n) &= \mathcal{H}\left[\sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m},\bar{n})\delta(m-\bar{m}, n-\bar{n})\right] \\ &= \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m},\bar{n})\mathcal{H}[\delta(m-\bar{m}, n-\bar{n})] \\ &= \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m},\bar{n})h(m-\bar{m}, n-\bar{n}) \\ &\doteq u(m,n) * h(m,n) \end{aligned}$$

Convolution Example

$$v(m,n) = u(m,n) * h(m,n) = \sum_{\bar{m}=-\infty}^{\infty} \sum_{\bar{n}=-\infty}^{\infty} u(\bar{m},\bar{n})h(m-\bar{m}, n-\bar{n})$$



Convolution of Continuous Functions



- Convolution

$$\begin{aligned} g(x,y) &= f(x,y) * h(x,y) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\bar{x},\bar{y})h(x-\bar{x},y-\bar{y})d\bar{x}d\bar{y} \end{aligned}$$

Fourier Transform

- Fourier transform pair

$$
\begin{aligned}
F(\xi_1, \xi_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(x\xi_1 + y\xi_2)]dxdy \\
f(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\xi_1, \xi_2)\exp[j2\pi(x\xi_1 + y\xi_2)]d\xi_1 d\xi_2
\end{aligned}
$$

- Fourier transform is an one-to-one mapping from function space to function space (information preserving).
- $\exp[j2\pi(x\xi_1 + y\xi_2)]$
  - large ξ_1 or $\xi_i \longrightarrow$ fast varying in x or y direction
  - (ξ_1, ξ_2): horizontal and vertical frequencies
  - $F(\xi_1, \xi_2)$: the amount of frequency component (ξ_1, ξ_2), which is contained within $f(x, y)$

Fourier Transform

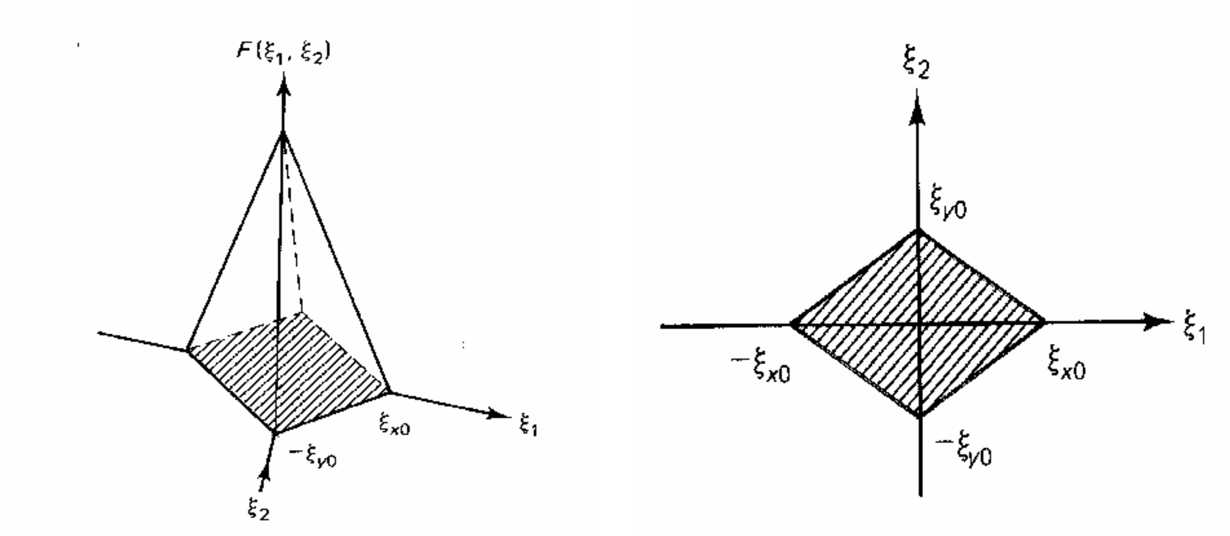
Properties of 2-D Fourier Transform

	Function	Its Fourier Transform
Linearity	$af(x,y)+bg(x,y)$	$aF(\xi_1,\xi_2)+bG(\xi_1,\xi_2)$
Convolution	$f(x,y)*g(x,y)$	$F(\xi_1,\xi_2)G(\xi_1,\xi_2)$
Multiplication	$f(x,y)g(x,y)$	$F(\xi_1,\xi_2)*G(\xi_1,\xi_2)$
Comb function	$\sum_{m,n=-\infty}^{\infty}\delta(x-m\Delta x, y-n\Delta y)$	$\frac{1}{\Delta x\Delta y}\sum_{k,l=-\infty}^{\infty}\delta(\xi_1-k\frac{1}{\Delta x}, \xi_2-l\frac{1}{\Delta y})$
Lowpass filter	$f(x,y)=\text{sinc}(\alpha x)\text{sinc}(\beta y)$	$F(\xi_1,\xi_2)=\begin{cases}\frac{1}{\alpha\beta}, & \lvert\xi_1\rvert<\frac{1}{2}\alpha, \lvert\xi_2\rvert<\frac{1}{2}\beta \\ 0, & \text{otherwise}\end{cases}$

Note: $\text{sinc}(x)=\dfrac{\sin\pi x}{\pi x}$

Sampling

- Bandlimited Images



Fourier Transform of a bandlimited function

Its region of support

- $f(x, y)$ is called bandlimited if

$$F(\xi_1, \xi_2) = 0 \quad \text{when } |\xi_1| > \xi_{x0} \text{ or } |\xi_2| > \xi_{y0}$$

Multiplying by Comb Function

- Sampling on a rectangular grid with spacing Δx, Δy
- Comb function

$$\text{comb}(x, y; \Delta x, \Delta y) \doteq \sum_{m,n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

- Sampled image $f_s(x, y)$

$$\begin{aligned} f_s(x, y) &= f(x, y)\text{comb}(x, y; \Delta x, \Delta y) \\ &= \sum_{m,n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y) \end{aligned}$$

Multiplying by Comb Function

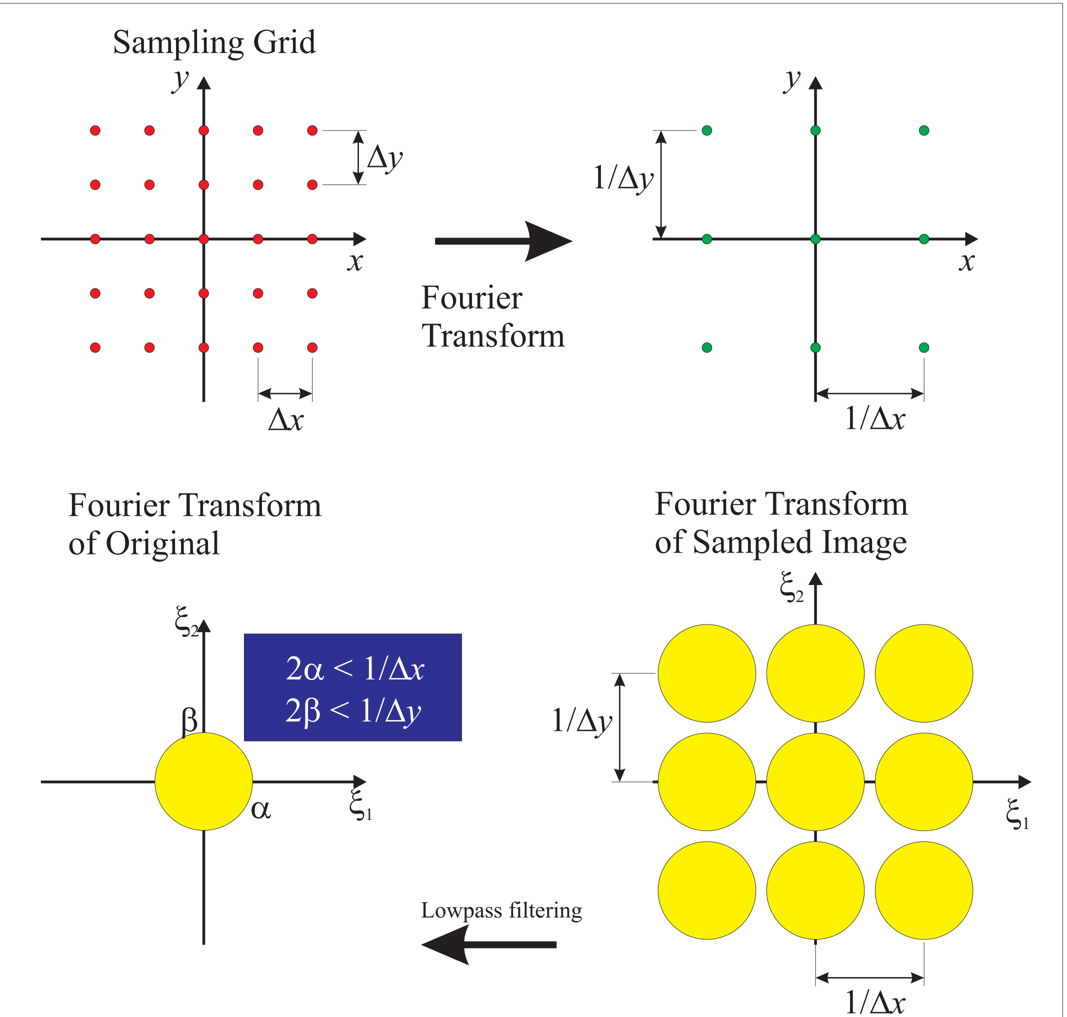
- Frequency Domain

$$\begin{aligned} F_s(\xi_1, \xi_2) &= F(\xi_1, \xi_2) * \mathcal{FOURIER}[\text{comb}(x, y; \Delta x, \Delta y)] \\ &= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} F(\xi_1, \xi_2) * \delta(\xi_1 - k\frac{1}{\Delta x}, \xi_2 - l\frac{1}{\Delta y}) \\ &= \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} F(\xi_1 - k\frac{1}{\Delta x}, \xi_2 - l\frac{1}{\Delta y}) \end{aligned}$$

- Periodic replication of original transform on a grid with spacing $(\frac{1}{\Delta x}, \frac{1}{\Delta y})$

Nyquist Sampling Theorem

$$F_s(\xi_1, \xi_2) = \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} F(\xi_1 - k\frac{1}{\Delta x}, \xi_2 - l\frac{1}{\Delta y})$$



- $(\Delta x, \Delta y)$ in space $\Rightarrow (\frac{1}{\Delta x}, \frac{1}{\Delta y})$ in frequency
- Nyquist Sampling Theorem

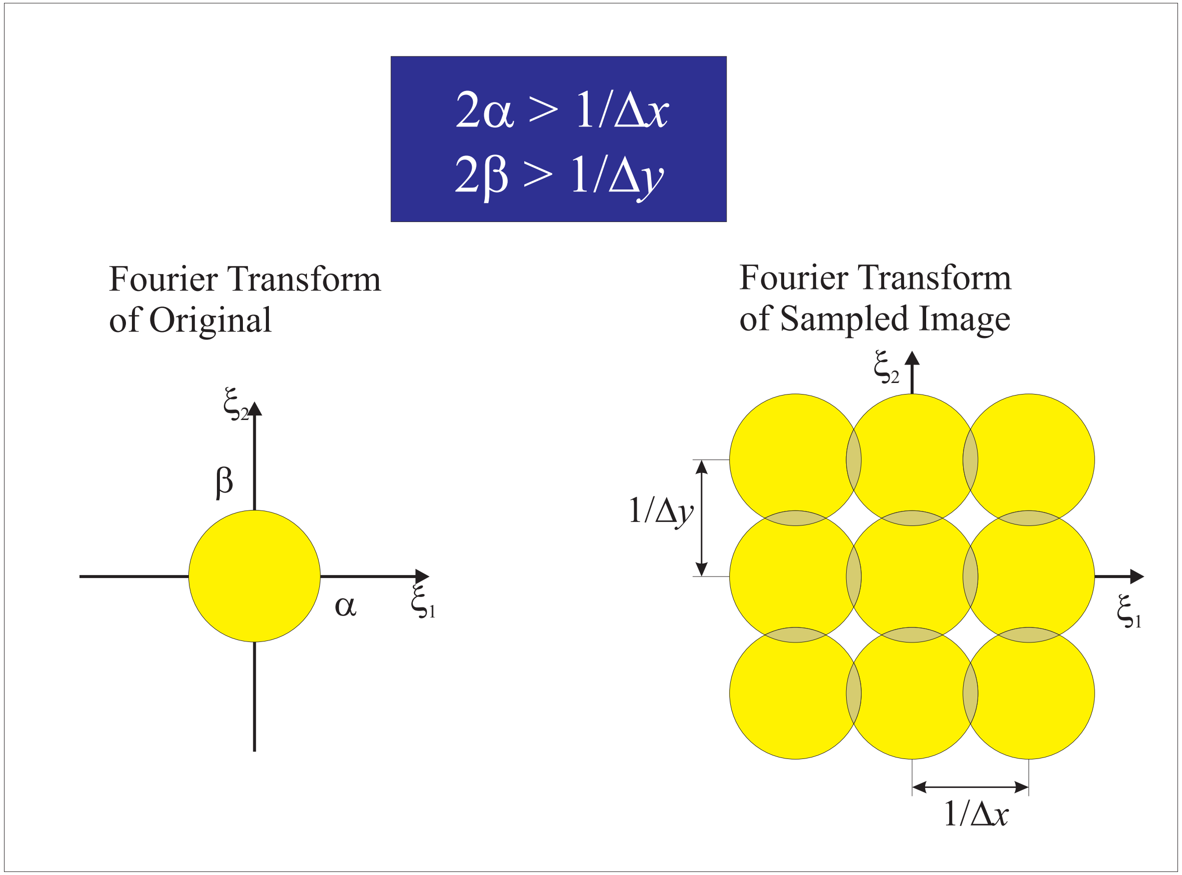
$$\xi_{xs} \doteq \frac{1}{\Delta x} > 2\alpha$$

$$\xi_{ys} \doteq \frac{1}{\Delta y} > 2\beta$$

- Perfect reconstruction filter

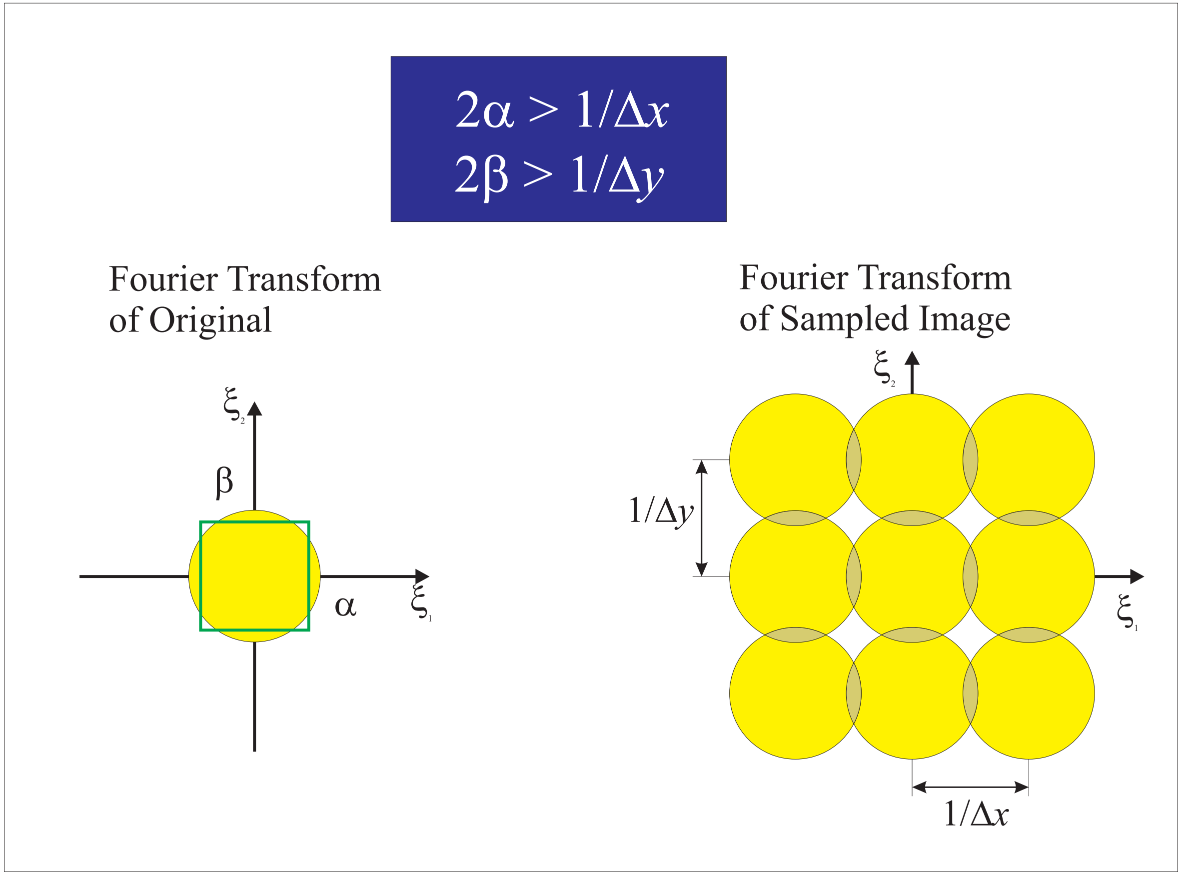
$$H(\xi_1, \xi_2) = \begin{cases} \Delta x \Delta y & |\xi_1| < \frac{1}{2\Delta x}, |\xi_2| < \frac{1}{2\Delta y} \\ 0 & \text{otherwise} \end{cases}$$

Aliasing



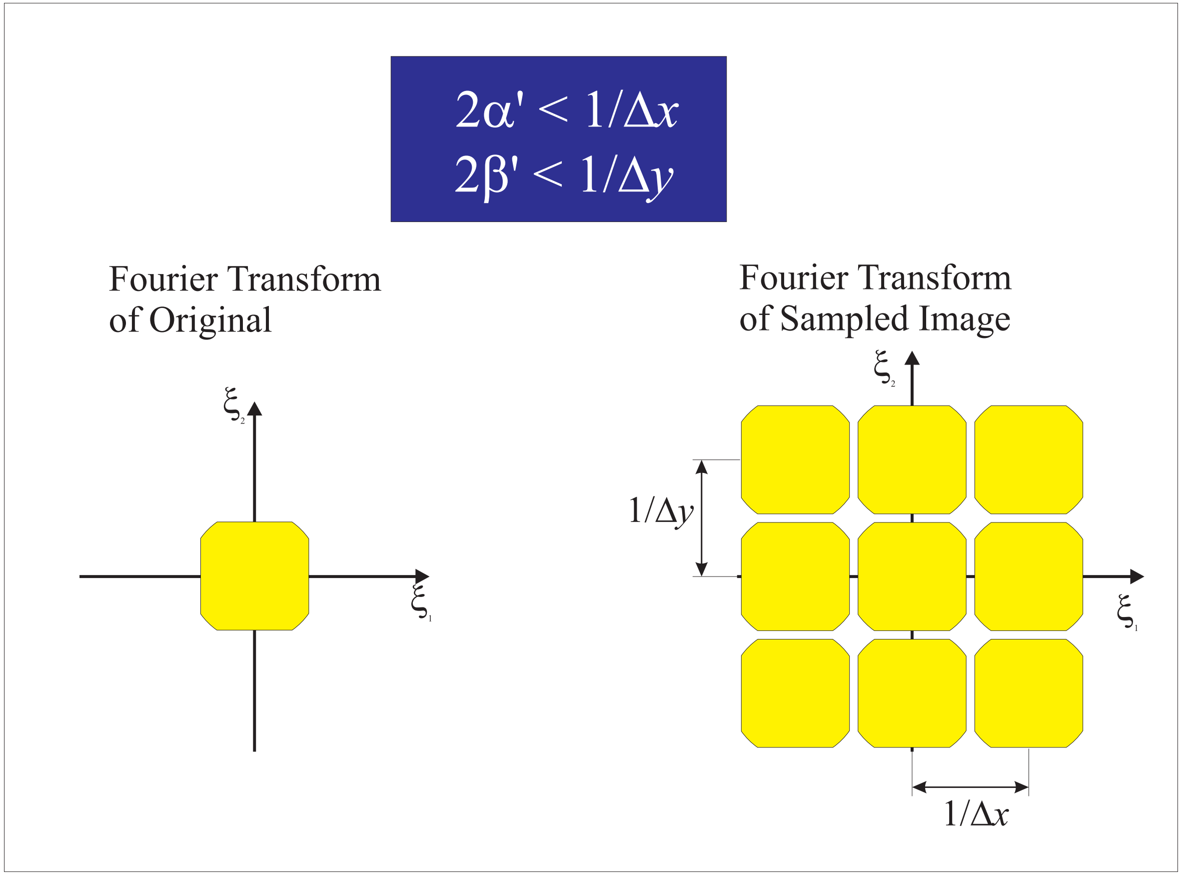
- Folding (addition) happens between two consecutive replicas of original transform
- Thus, original transform (and image) cannot be reconstructed
- For example, let $a + b = c$. Can you recover a and b from c?

Pre-filtering



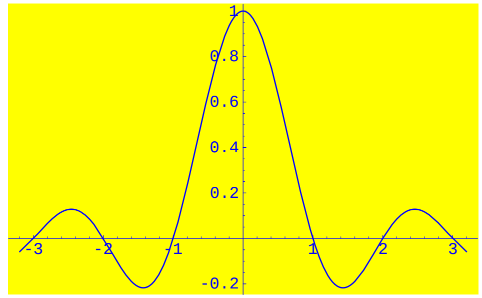
- Before sampling, use a lowpass filter with cutoff frequencies (c_x, c_y)
  - $c_x < \frac{1}{2\Delta_x}$
  - $c_y < \frac{1}{2\Delta_y}$

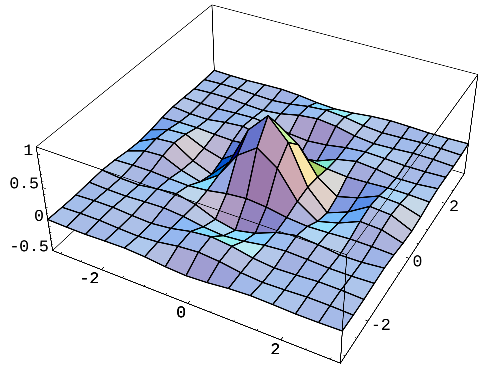
Pre-filtering



- No aliasing due to pre-filtering
- Loss of high frequency information
- Pre-filtering is used in some applications because aliasing is unpredictable but the information loss is predictable.

Perfect Reconstruction Filter





- $$f_s(x,y) = \sum_{m,n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y)$$

  $$\Rightarrow \quad F_s(\xi_1, \xi_2) = \frac{1}{\Delta x \Delta y} \sum_{k,l=-\infty}^{\infty} F(\xi_1 - k\frac{1}{\Delta x}, \xi_2 - l\frac{1}{\Delta y})$$

- $$H(\xi_1, \xi_2) = \begin{cases} \Delta x \Delta y & |\xi_1| < \frac{1}{2\Delta x}, |\xi_2| < \frac{1}{2\Delta y} \\ 0 & \text{otherwise} \end{cases}$$

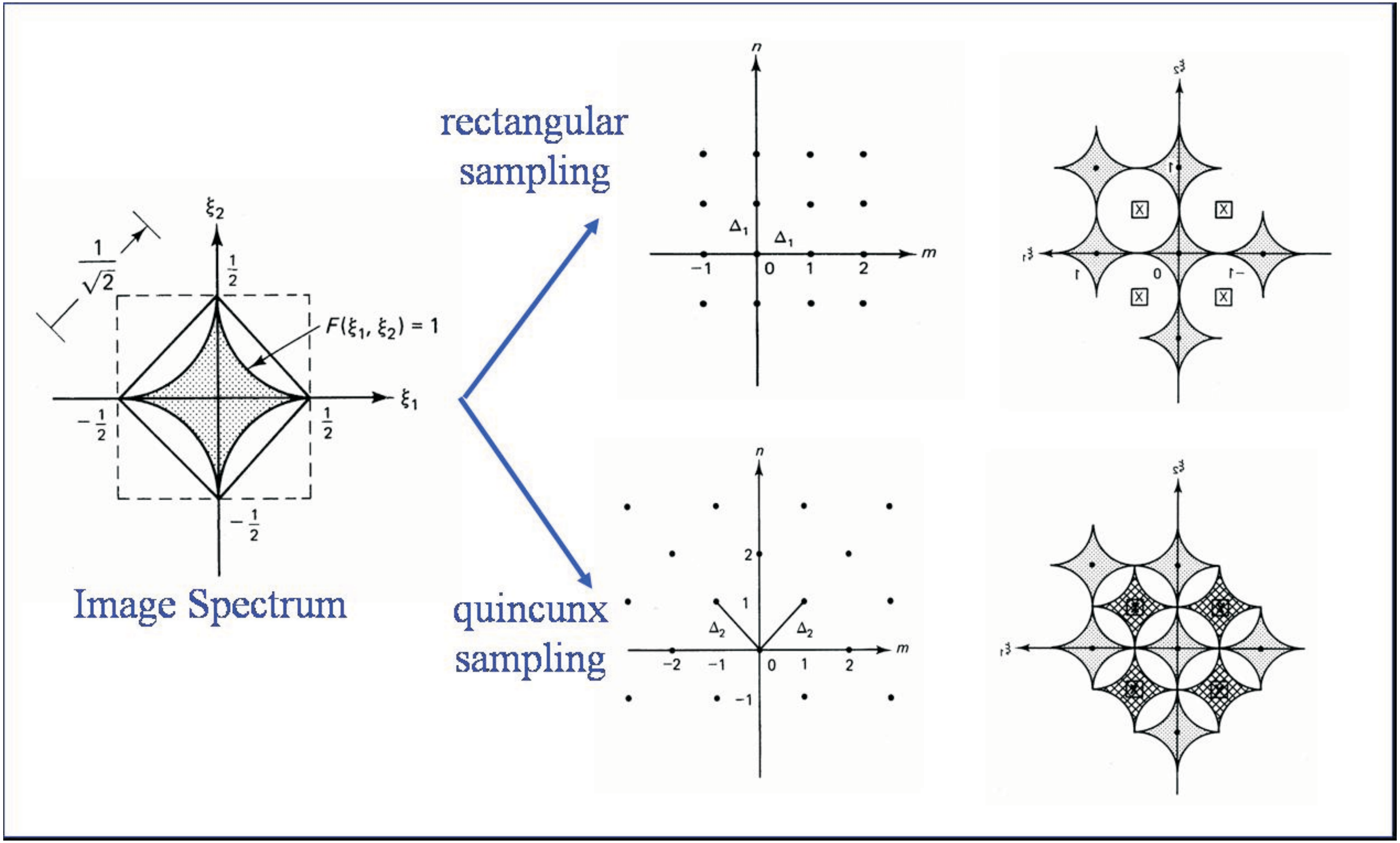
  $$\Rightarrow \quad h(x,y) = \mathrm{sinc}(\frac{x}{\Delta x})\mathrm{sinc}(\frac{y}{\Delta y})$$

- $$f(x,y) = f_s(x,y) * h(x,y)$$

  $$= \sum_{m,n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y) * \mathrm{sinc}(\frac{x}{\Delta x})\mathrm{sinc}(\frac{y}{\Delta y})$$

  $$= \sum_{m,n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\mathrm{sinc}(\frac{x - m\Delta x}{\Delta x})\mathrm{sinc}(\frac{y - n\Delta y}{\Delta y})$$

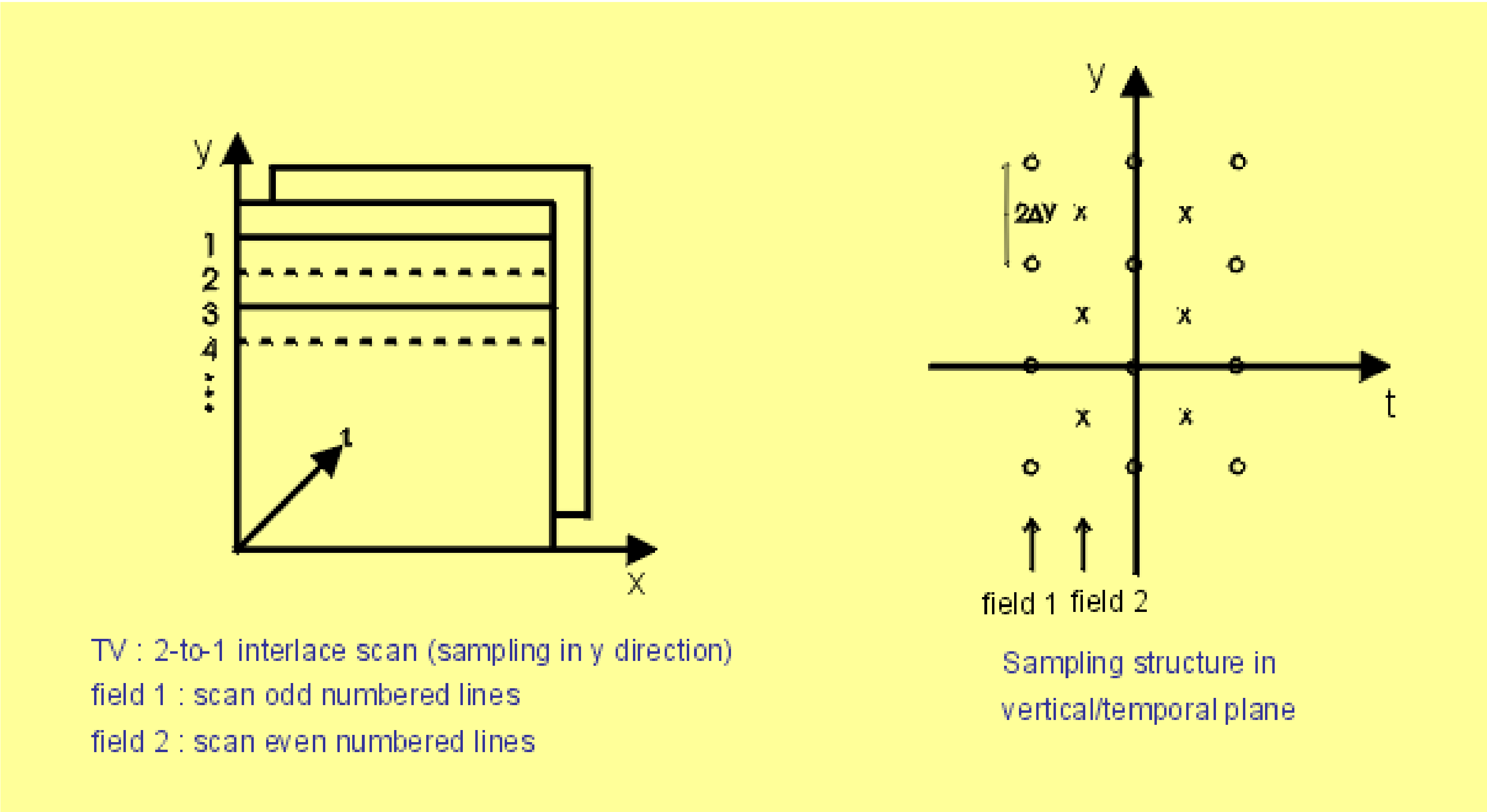
Nonrectangular Grid Sampling and Interlacing



Sampling resolution can be reduced by a factor of 2 by using quincunx sampling lattice

Nonrectangular Grid Sampling and Interlacing

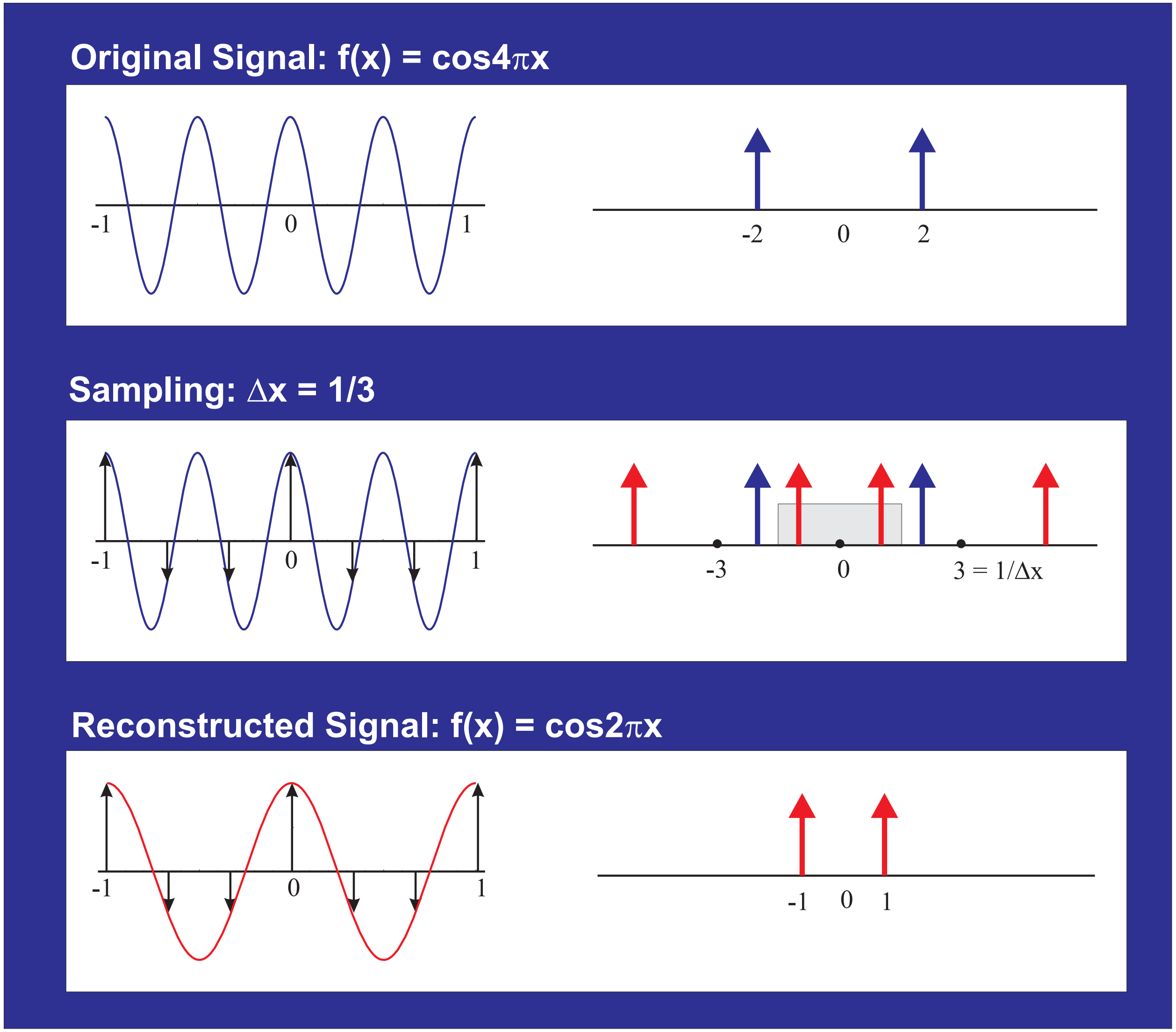
- Interlacing scan in TV system



Practical Limitations in Sampling and Reconstruction

- Real image signal - not bandlimited
  - aliasing is inevitable
- Low-pass filtering before sampling
  - can alleviate aliasing
  - but attenuate higher spatial frequency information (image blurring)
- Non-ideal reconstruction filter (display or printer)

Example of Aliasing



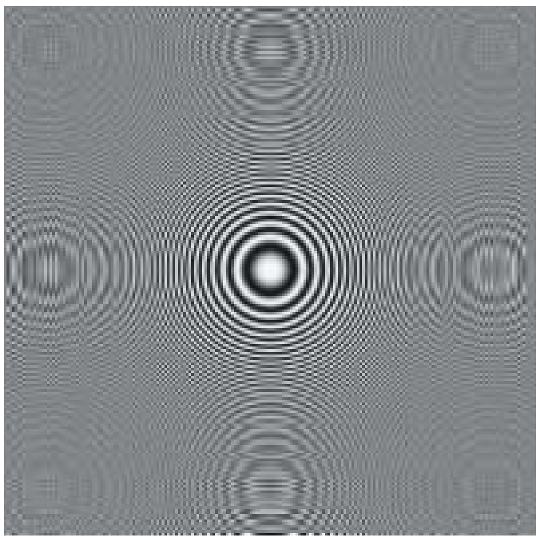
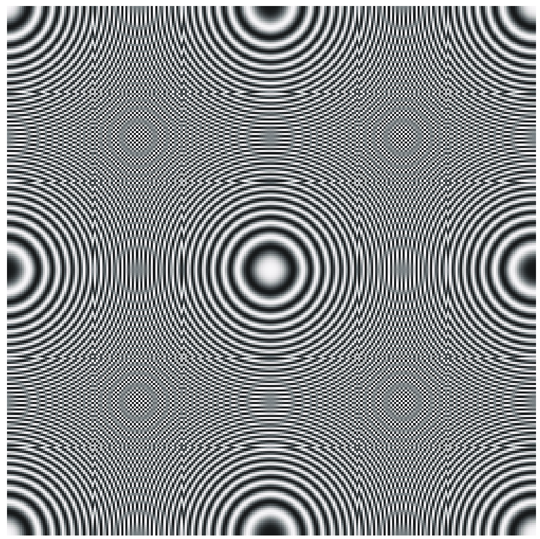
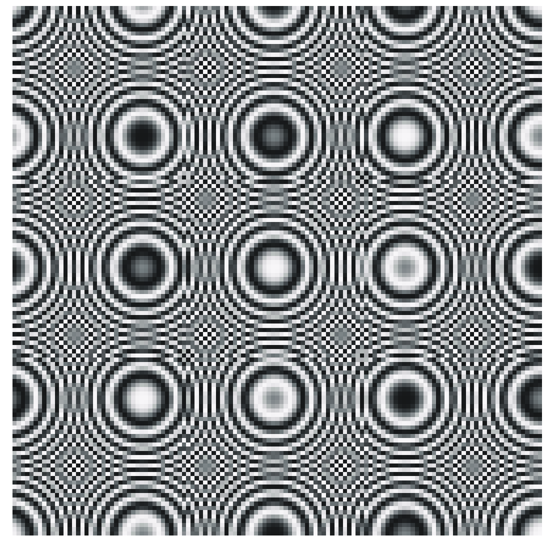
- Bandwidth of the original signal = 2
- Nyquist sampling rate

$$\frac{1}{\Delta x} > 2 \cdot 2 = 4, \quad \Delta x < \frac{1}{4}$$

- By mistake, one assume that the bandwidth is less than 1.5 and use the sampling distance $\Delta x = \frac{1}{3}$
- Reconstruction filter yields frequencies higher than 1.5
- Aliasing causes totally wrong reconstruction

Example of Aliasing (Cont'd)

- Cocentric circles

High — Low

Sampling frequency

Low-pass Filtering

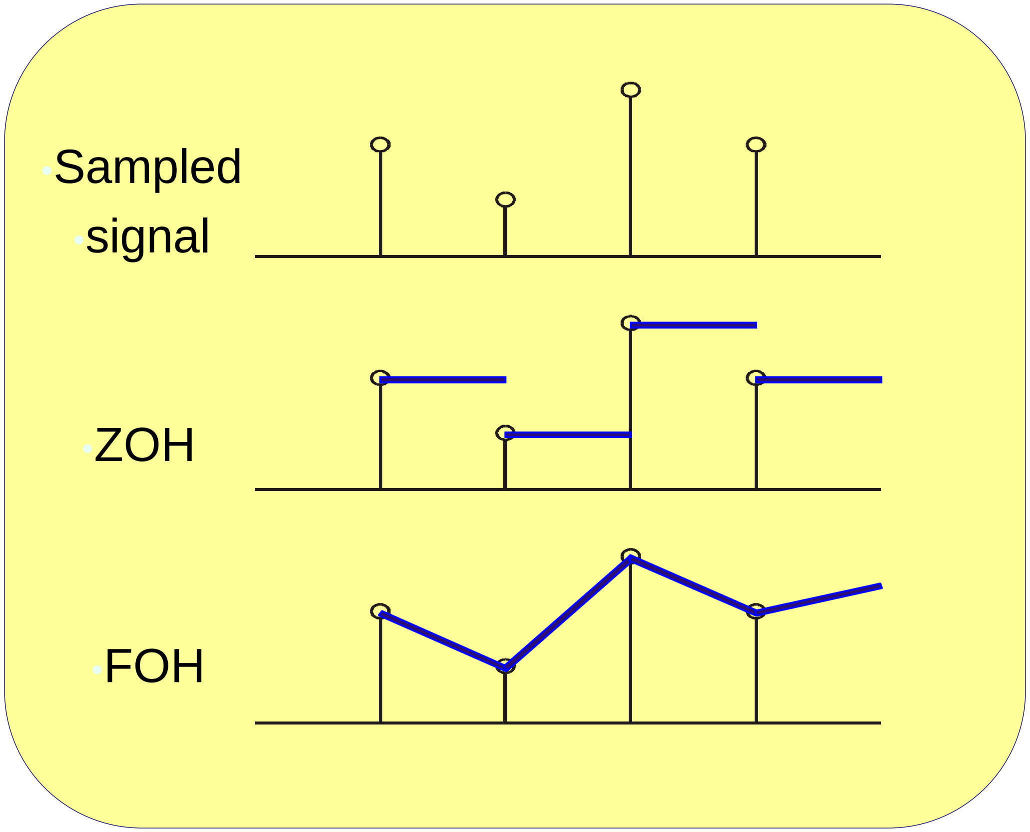
- Aliasing can be alleviated by low-passing original image before sampling

With pre-filtering

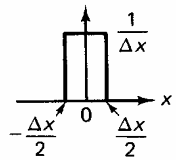
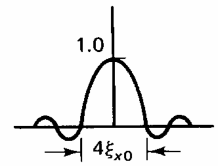
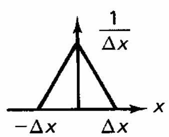
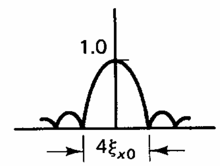
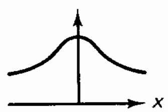
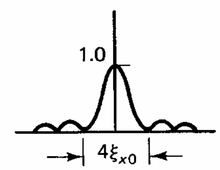
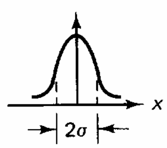
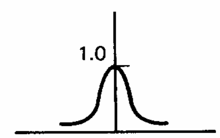
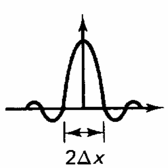
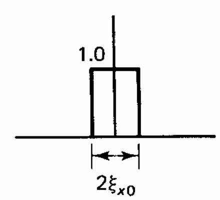
Without pre-filtering

Practical Interpolation Filter

- Ideal filter – sinc function
  - not implemetable
  - infinite duration
  - negative lobe
- Repetition or zero-order holding (ZOH)
- Linear interpolation or first-order holding (FOH)
- Quadratic
- Cubic spline
- Gaussian

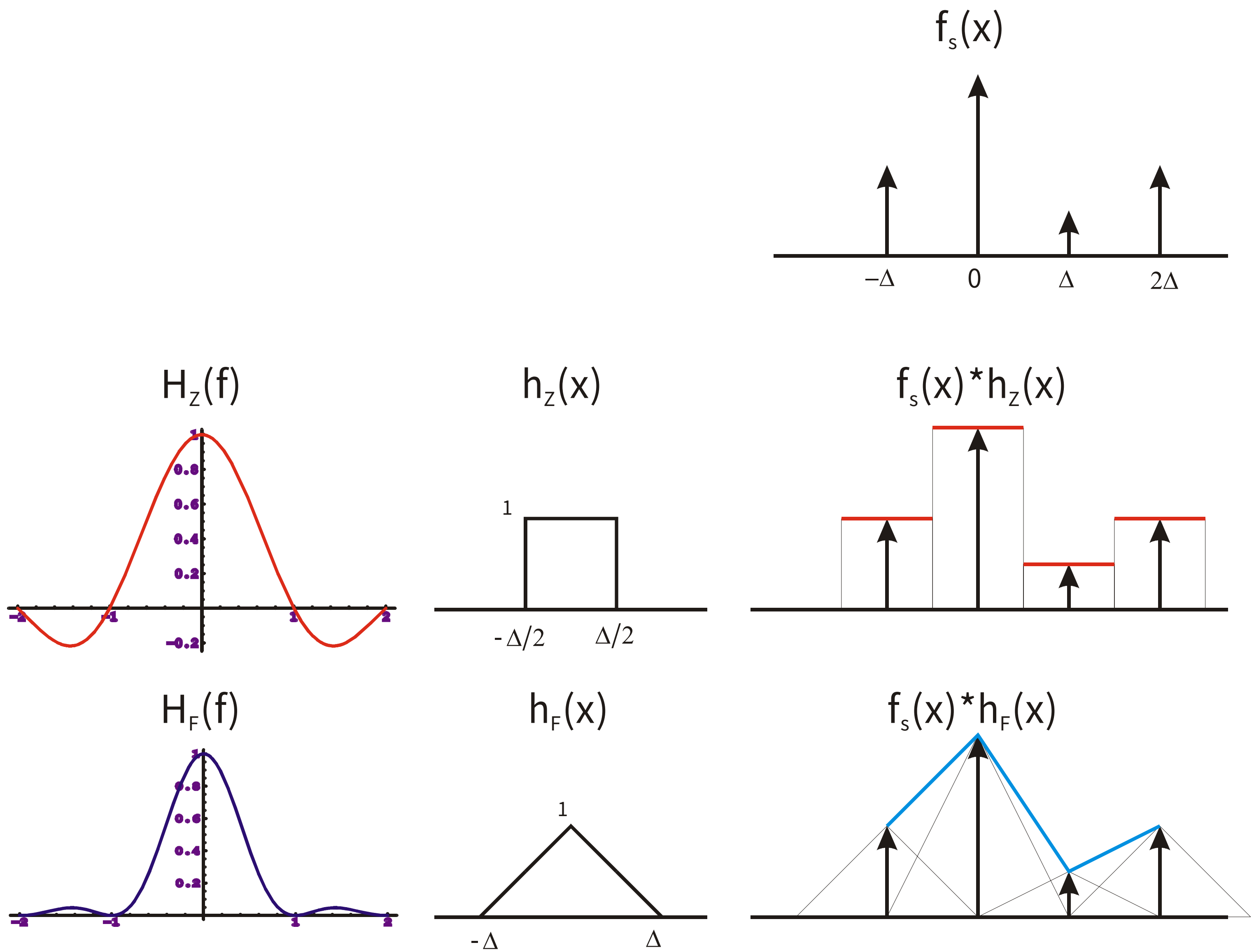


Practical Interpolation Filter

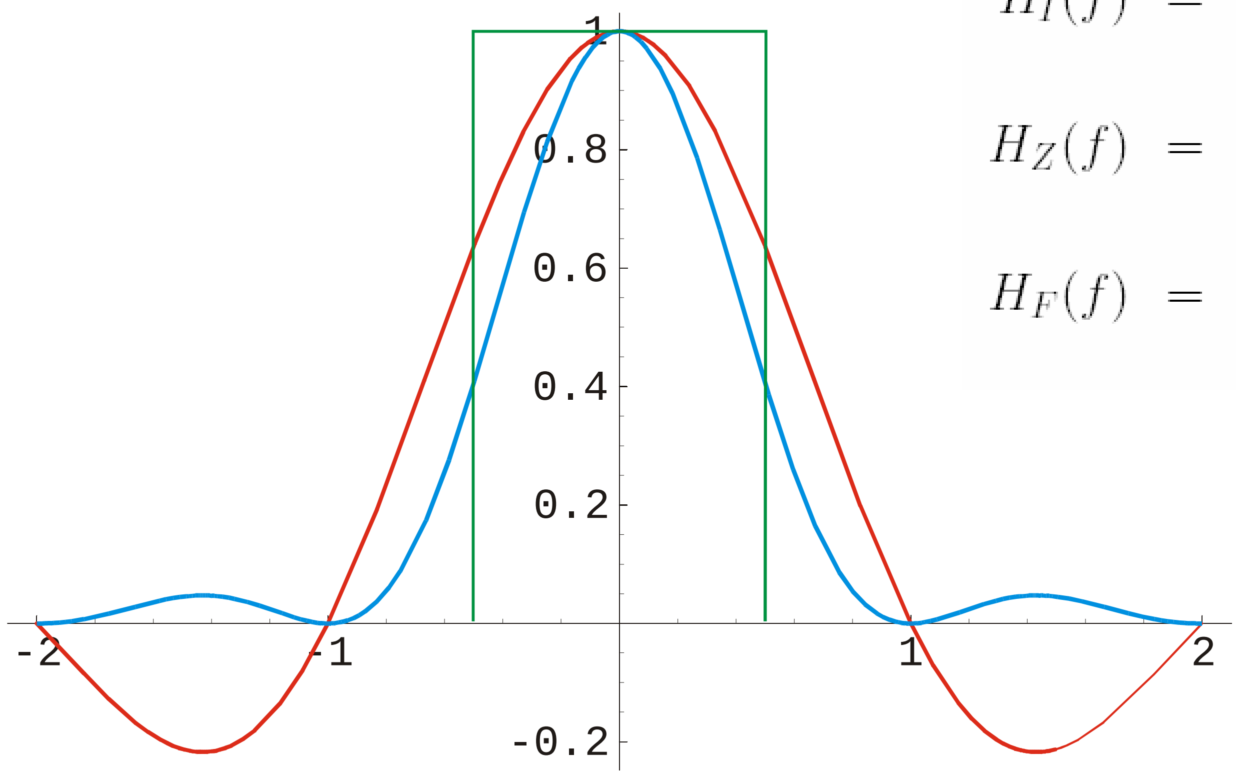
One-dimensional interpolation function	Diagram	Definition $p(x)$	Two-dimensional interpolation function $p_d(x, y) = p(x)p(y)$	Frequency response $P_d(\xi_1, \xi_2)$	$P_d(\xi_1, 0)$
ZOH		$\frac{1}{\Delta x}\,\text{rect}\left(\frac{x}{\Delta x}\right)$	$p_o(x)p_o(y)$	$\text{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\text{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	
FOH		$\frac{1}{\Delta x}\,\text{tri}\left(\frac{x}{\Delta x}\right)$ $p_o(x) \circledast p_o(x)$	$p_1(x)p_1(y)$	$\left[\text{sinc}\left(\frac{\xi_1}{2\xi_{x0}}\right)\text{sinc}\left(\frac{\xi_2}{2\xi_{y0}}\right)\right]^2$	
NOH		$p_o(x) \circledast \cdots \circledast p_o(x)$ n convolutions	$p_n(x)p_n(y)$	$\left[\text{sinc}\left(\frac{\xi_1}{\xi_{x0}}\right)\text{sinc}\left(\frac{\xi_2}{\xi_{y0}}\right)\right]^{n+1}$	
Gaussian		$\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{x^2}{2\sigma^2}\right]$	$\frac{1}{2\pi\sigma^2}\exp\left[-\frac{(x^2+y^2)}{2\sigma^2}\right]$	$\exp\left[-2\pi^2\sigma^2(\xi_1^2+\xi_2^2)\right]$	
Ideal		$\frac{1}{\Delta x}\,\text{sinc}\left(\frac{x}{\Delta x}\right)$	$\frac{1}{\Delta x \Delta y}\,\text{sinc}\left(\frac{x}{\Delta x}\right)\text{sinc}\left(\frac{x}{\Delta y}\right)$	$\text{rect}\left(\frac{\xi_1}{2\xi_{x0}}\right)\text{rect}\left(\frac{\xi_2}{2\xi_{y0}}\right)$	

less aliasing, more information loss

ZOH vs FOH


$f_s(x)$
$-\Delta$
0
Δ
2Δ
$H_Z(f)$
$h_Z(x)$
1
$-\Delta/2$
$\Delta/2$
$f_s(x)*h_Z(x)$
$H_F(f)$
$h_F(x)$
1
$-\Delta$
Δ
$f_s(x)*h_F(x)$

ZOH vs FOH



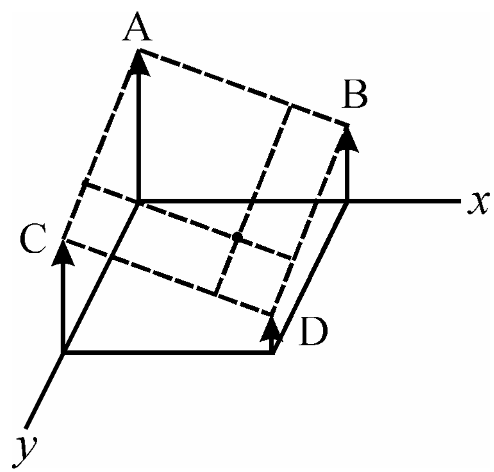
$$H_I(f) = \begin{cases} \Delta & \text{if } |f| < \frac{1}{2\Delta} \\ 0, & \text{otherwise} \end{cases}$$

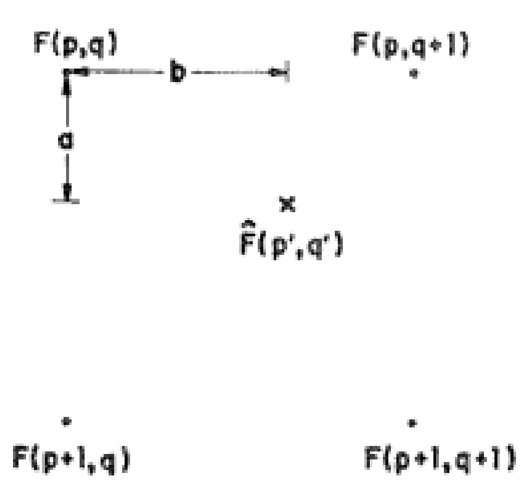
$$H_Z(f) = \frac{\sin(\pi f \Delta)}{\pi f}$$

$$H_F(f) = \frac{1}{\Delta}\left(\frac{\sin(\pi f \Delta)}{\pi f}\right)^2$$

Practical Interpolation Filter

- FOH is a good compromise between information loss and aliasing suppression
- 2D FOH – bilinear interpolation





$$\hat{F}(p',q') = (1-a)[(1-b)F(p,q) + bF(p,q+1)] + a[(1-b)F(p+1,q) + bF(p+1,q+1)]$$

Example – Step 1. Sampling

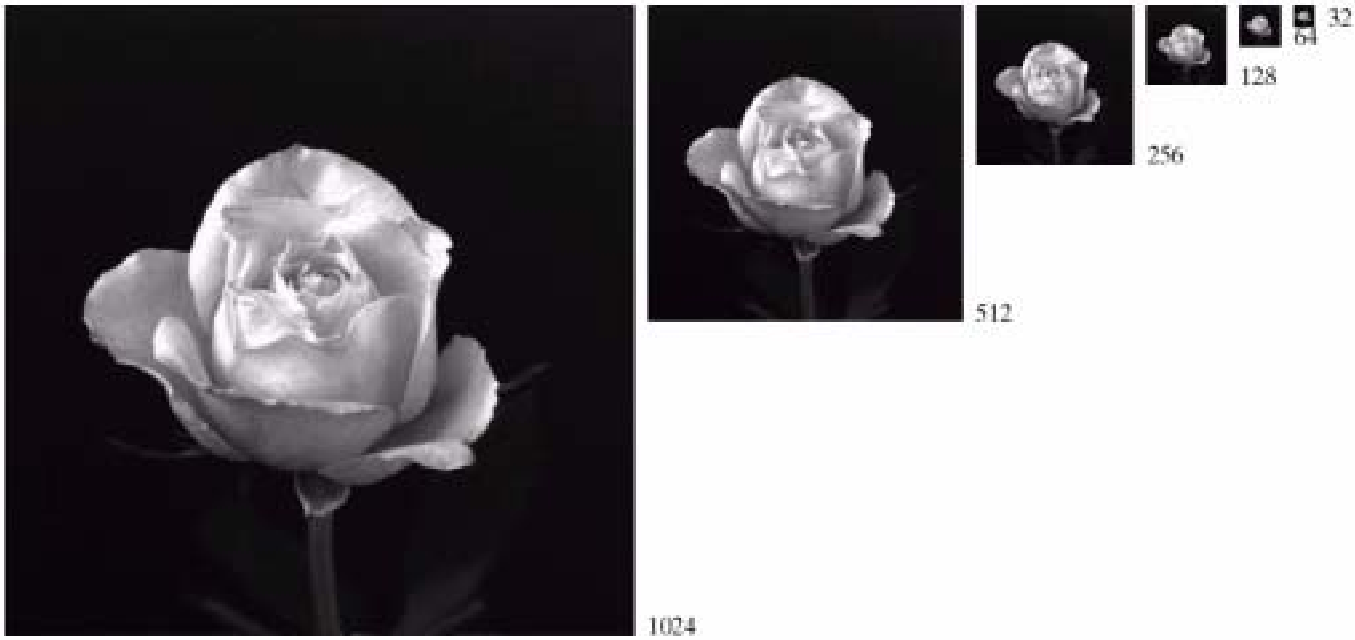
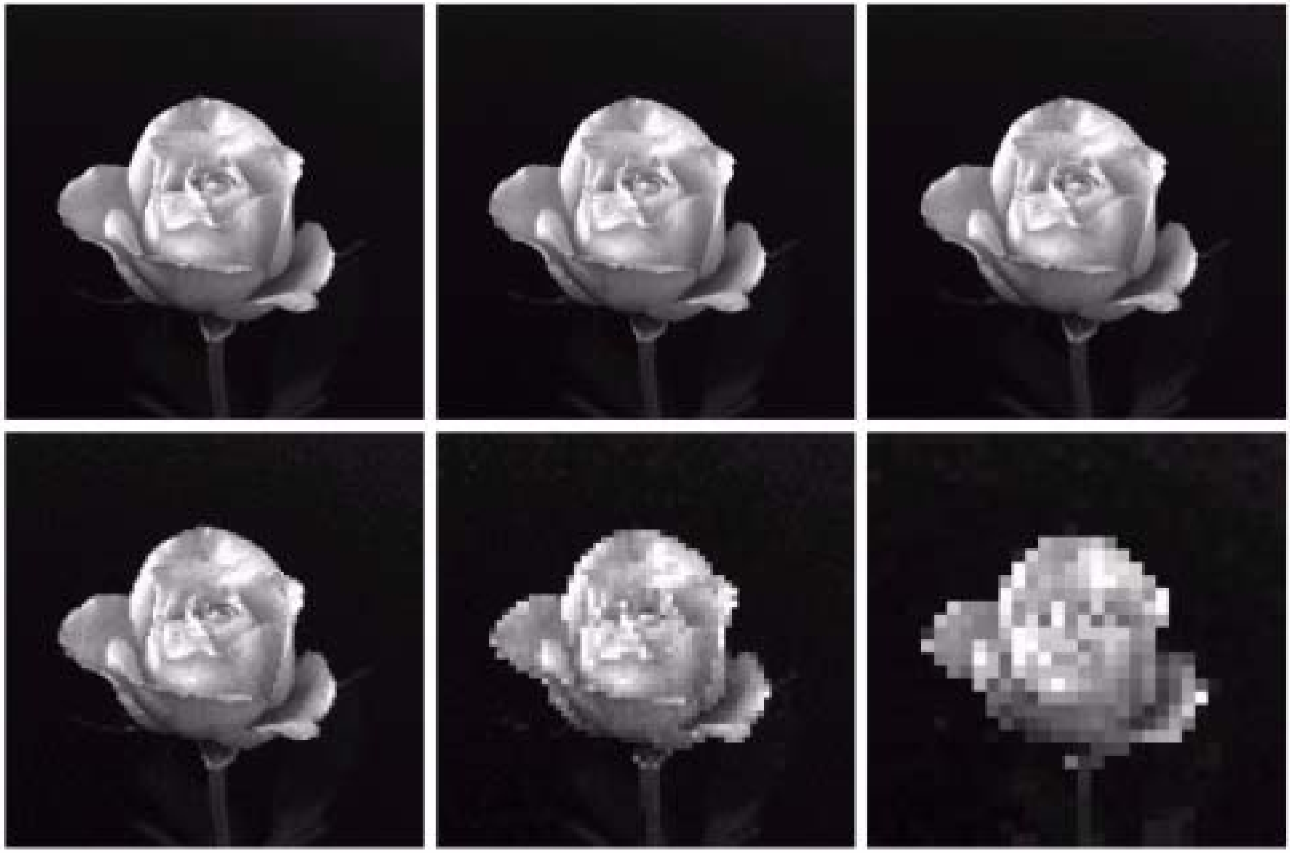


FIGURE 2.19 A 1024 × 1024, 8-bit image subsampled down to size 32 × 32 pixels. The number of allowable gray levels was kept at 256.

Example – Step 2. Interpolation (ZOH)



a b c
d e f

FIGURE 2.20 (a) 1024 × 1024, 8-bit image. (b) 512 × 512 image resampled into 1024 × 1024 pixels by row and column duplication. (c) through (f) 256 × 256, 128 × 128, 64 × 64, and 32 × 32 images resampled into 1024 × 1024 pixels.

Example –ZOH vs. FOH

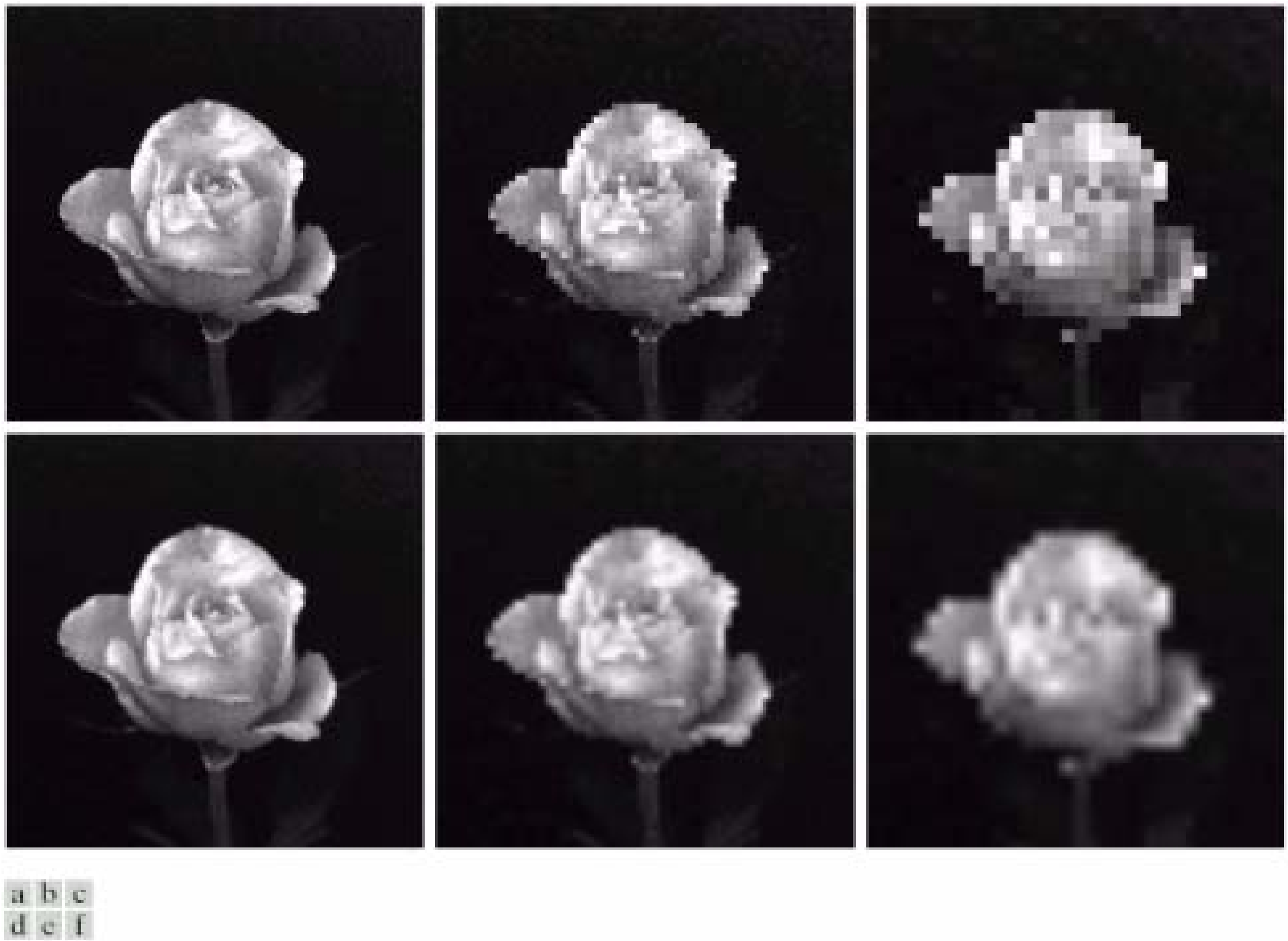


FIGURE 2.25 Top row: images zoomed from 128 × 128, 64 × 64, and 32 × 32 pixels to 1024 × 1024 pixels, using nearest neighbor gray-level interpolation. Bottom row: same sequence, but using bilinear interpolation.

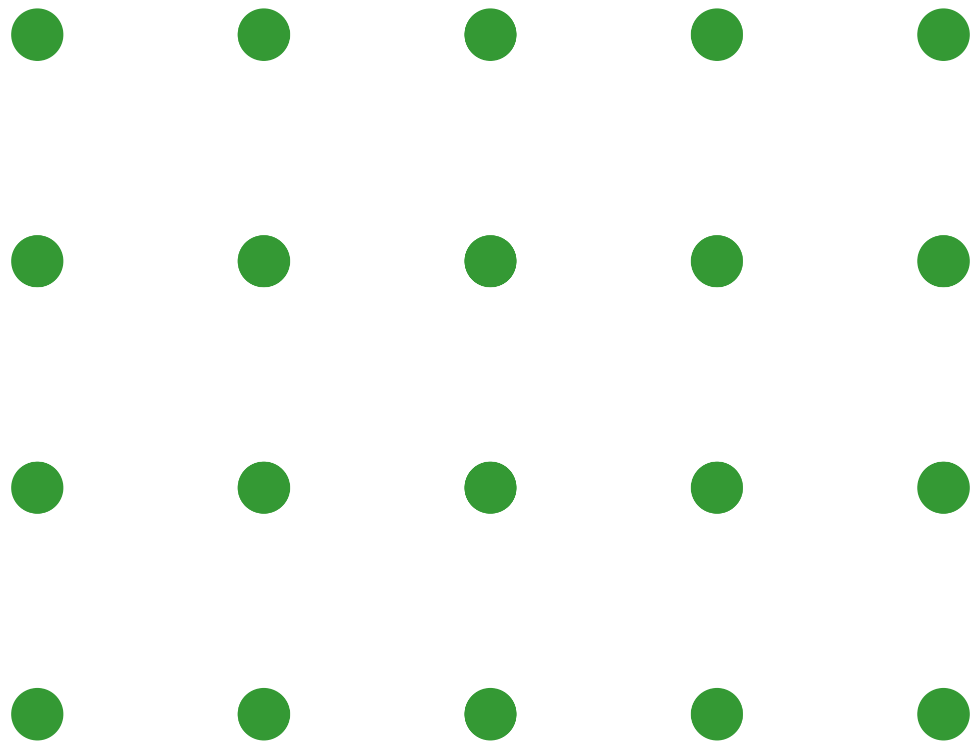
Zooming and Shrinking Digital Images

- Resolution conversion from (M_1,N_1) to (M_2,N_2)
  - M_1/M_2 and N_1/N_2 can be arbitrary

1. Digital-to-Analog conversion (e.g. bilinear interpolation)
2. Resampling

Zooming Case

$M_1/M_2 = N_1/N_2 = 2/3$



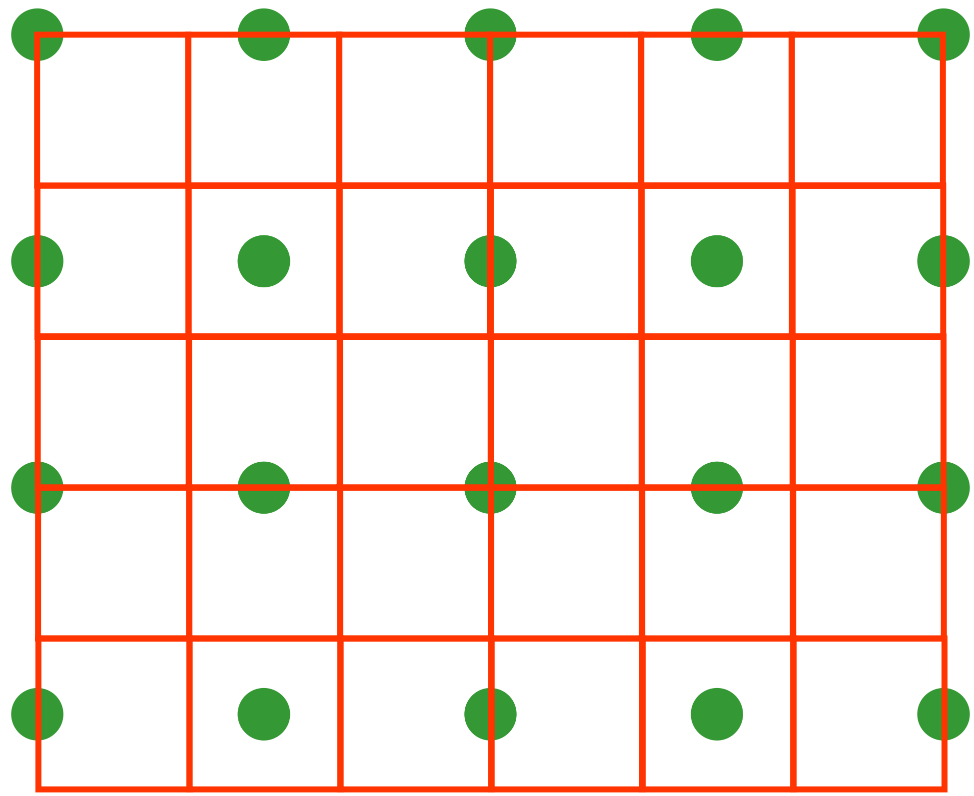
Zooming and Shrinking Digital Images

- Resolution conversion from (M_1,N_1) to (M_2,N_2)
  - M_1/M_2 and N_1/N_2 can be arbitrary

1. Digital-to-Analog conversion (e.g. bilinear interpolation)
2. Resampling

Zooming Case

$M_1/M_2 = N_1/N_2 = 2/3$



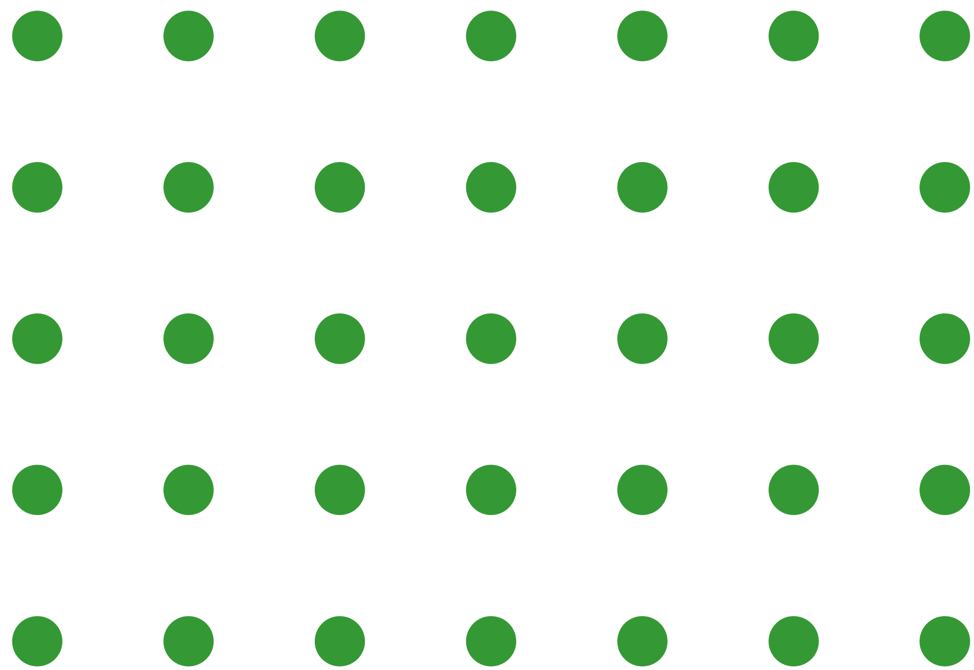
Zooming and Shrinking Digital Images

- Resolution conversion from (M_1,N_1) to (M_2,N_2)
  - M_1/M_2 and N_1/N_2 can be arbitrary

1. Digital-to-Analog conversion   (e.g. bilinear interpolation)
2. Resampling

Shrinking Case

$M_1/M_2 = N_1/N_2 = 3/2$



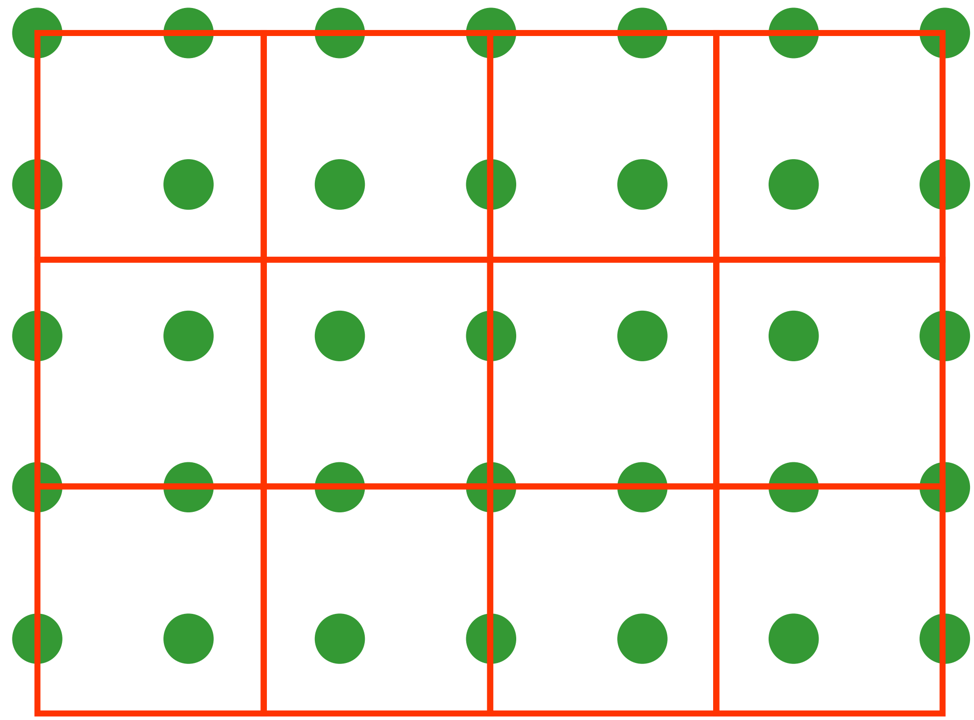
Zooming and Shrinking Digital Images

- Resolution conversion from (M_1,N_1) to (M_2,N_2)
  - M_1/M_2 and N_1/N_2 can be arbitrary

1. Digital-to-Analog conversion (e.g. bilinear interpolation)
2. Resampling

Shrinking Case

$M_1/M_2 = N_1/N_2 = 3/2$

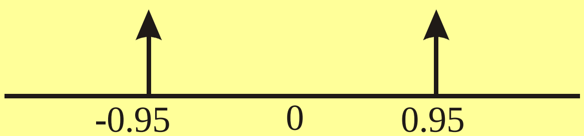


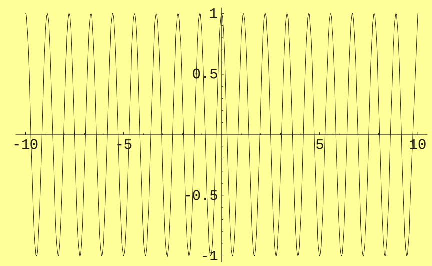
Moiré Effect

- Moiré pattern arises if
  - image contains periodicities that are close to half the sampling frequencies, and
  - reconstruction filter cutoff extends beyond the ideal low-pass filter cutoff
  - e.g. small display spot size => large cutoff frequency

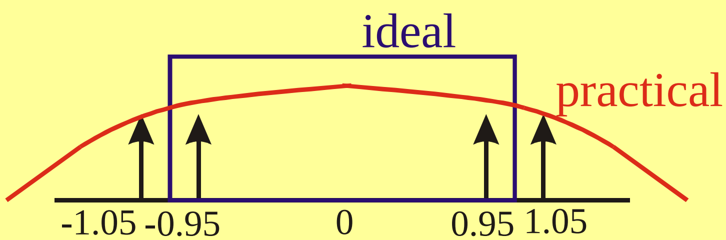
Moiré Effect – 1D example

Original signal $f(x) = \cos 2\pi(0.95)x$

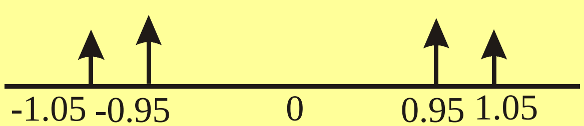


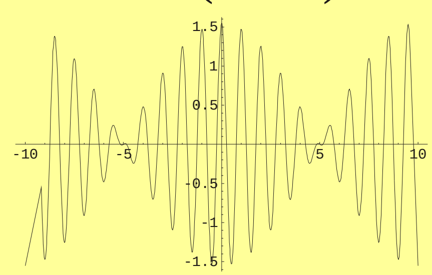


Sampling: $\Delta x = 1/2$



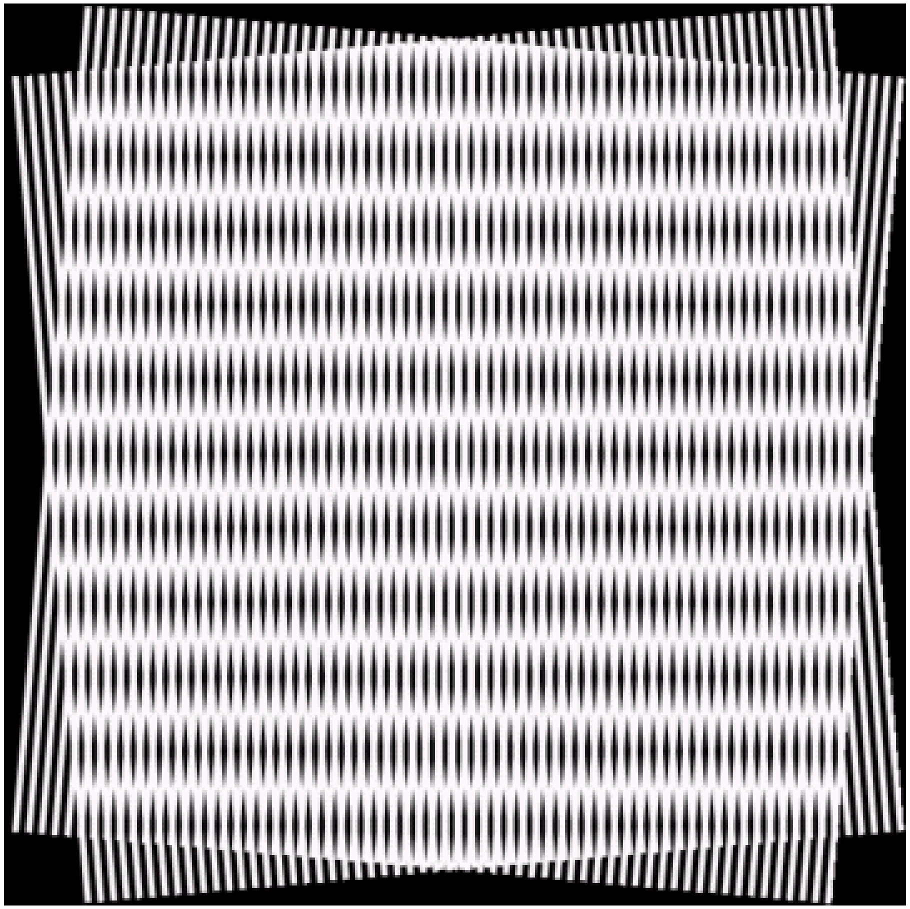
Reconstruction $g(x) = 0.8 \cos 2\pi(0.95)x + 0.75 \cos 2\pi(1.05x)$





- Amplitude modulated signal

Moiré Effect – 2D Example



corrugated tin loop

FIGURE 2.24 Illustration of the Moiré pattern effect.