Digital Signal Processing

Chap 4. Sampling of Continuous-Time Signals

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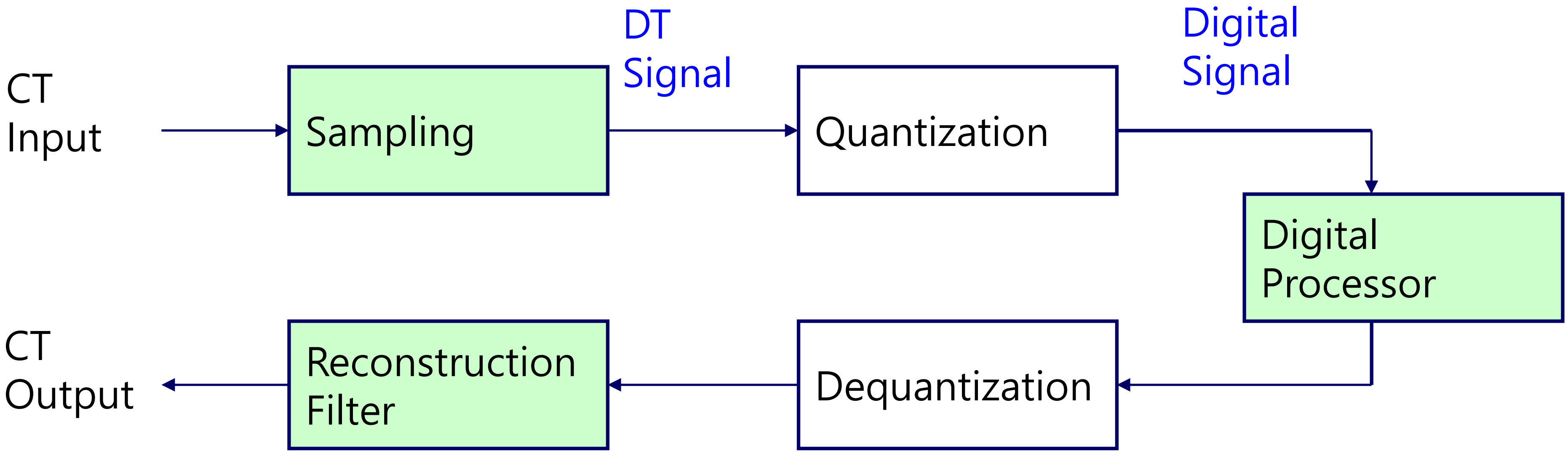
Digital Processing of Continuous-Time Signals

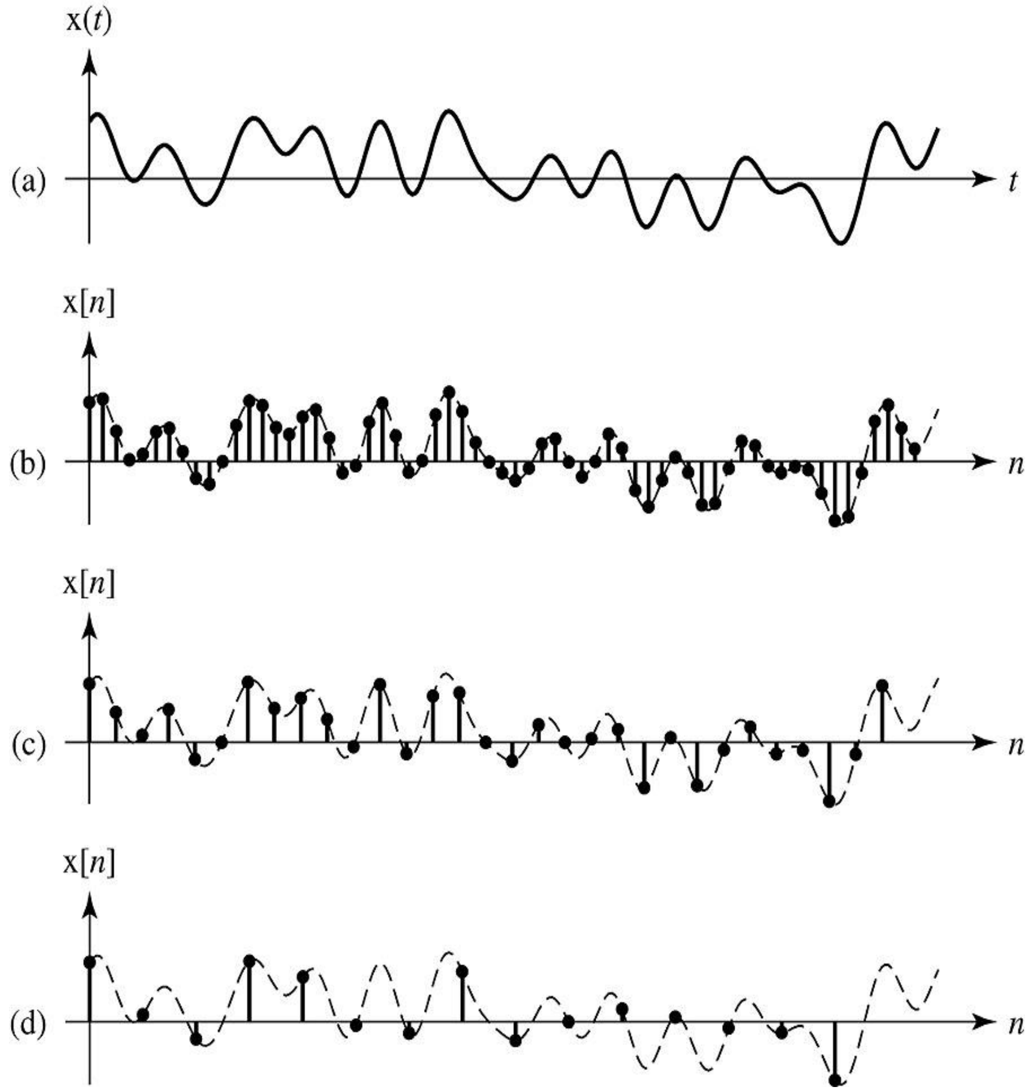
- Digital processing of a CT signal involves three basic steps
  1. Conversion of the CT signal into a DT signal
  2. Processing of the DT signal
  3. Conversion of the processed DT signal back into a CT signal

Conceptual Block diagram



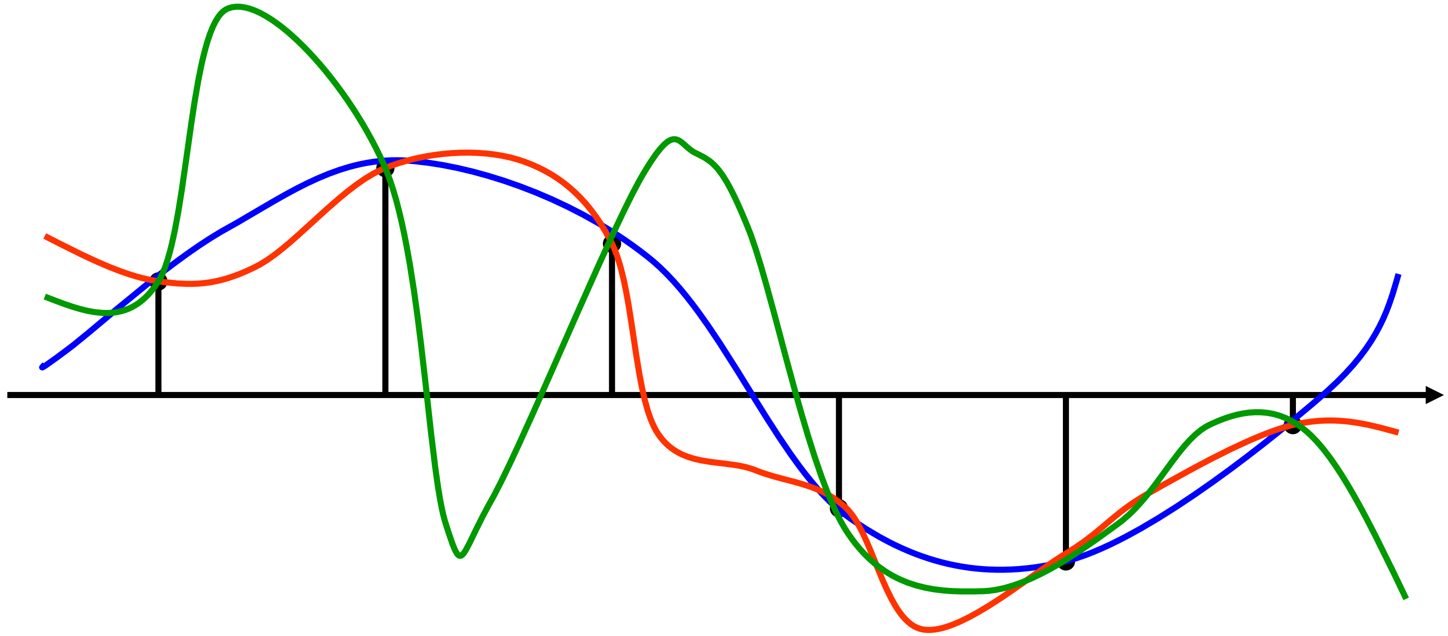
Sampling

Sampling



- Sampling is a procedure to extract a DT signal from a CT signals
- (b), (c), (d) are obtained by sampling (a)
- Is (b) enough to represent (a)?
- What is the adequate sampling rate to represent a given CT signal without information loss?

In general, DT signal cannot represent CT signal perfectly



Are these sample enough to reconstruct the original blue curve?

Continuous-Time Fourier Transform

- CTFT Formulae
  - Forward transform

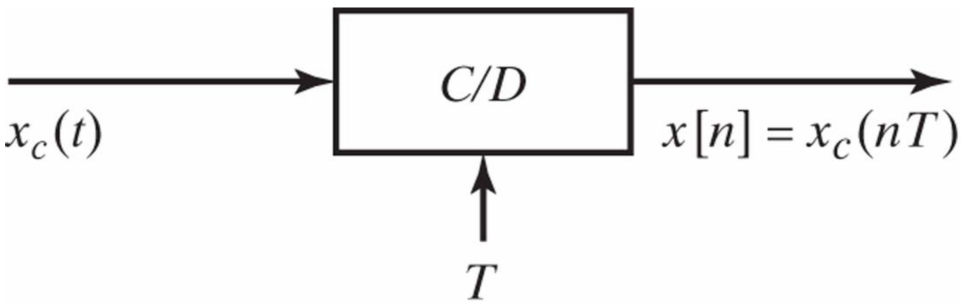
$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

  - Inverse transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

- We will use a number of properties of CTFT without proofs
  - They are studied in the course *Signals and Systems*

Periodic Sampling



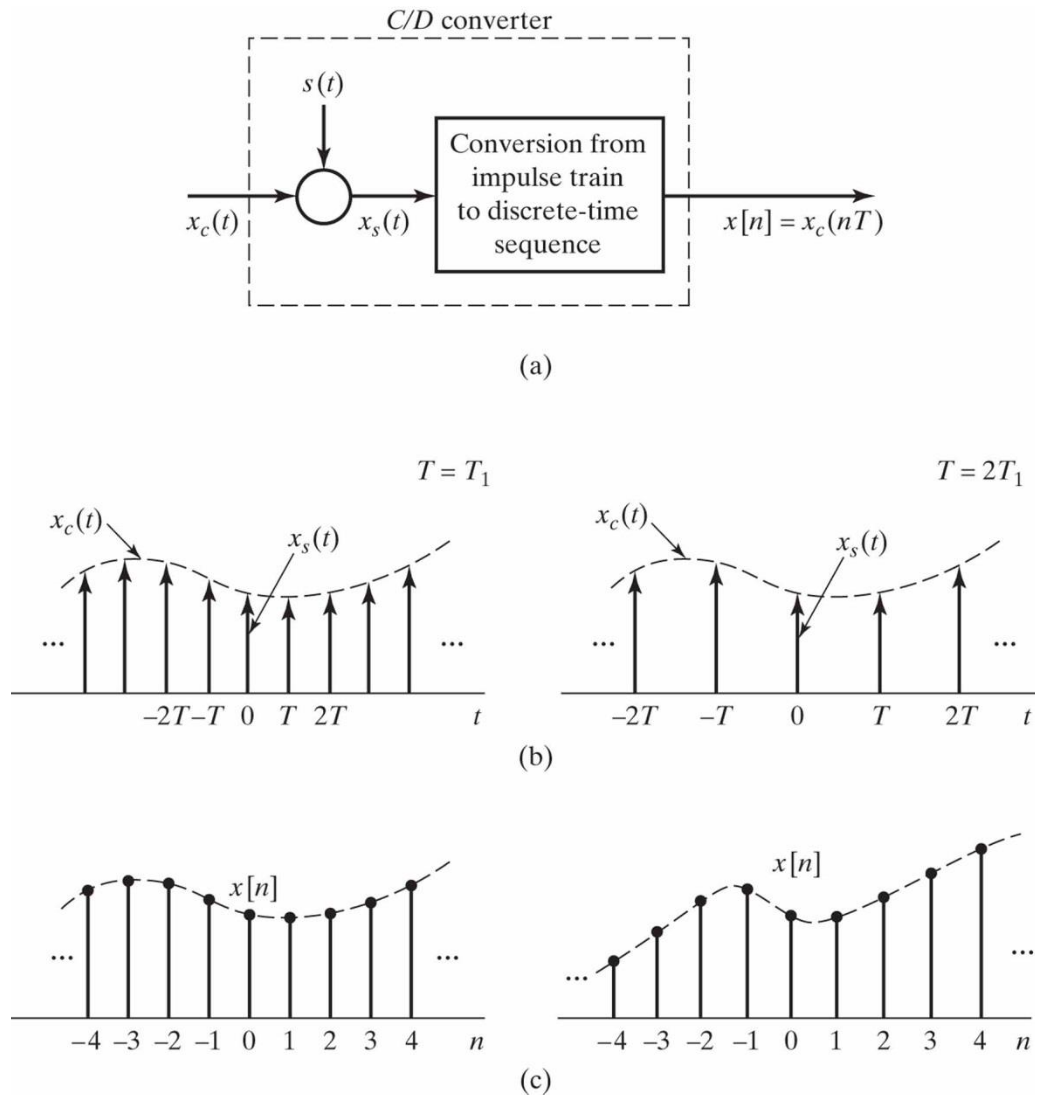
- C/D (continuous-time to discrete-time) converter
- $x[n] = x_c(nT), \ -\infty < n < \infty.$
  - T : sampling period
  - $\Omega_s = \frac{2\pi}{T}$ (or $f_s = \frac{1}{T}$) : sampling frequency

Periodic Sampling

- Conceptually, it is easier to introduce an impulse train for the C/D conversion

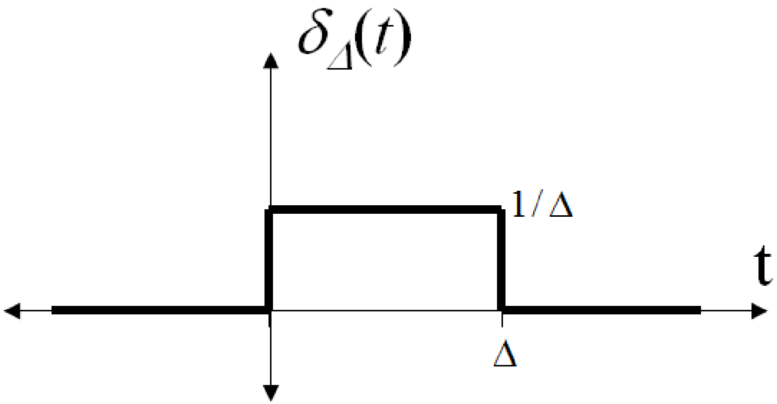
- $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- $x_s(t) = x_c(t)s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$

- $x_s(t)$ and $x[n]$ have the same information
  - Given $x_s(t)$, we can make $x[n]$.
  - Given $x[n]$, we can make $x_s(t)$.



- Approximated unit impulse

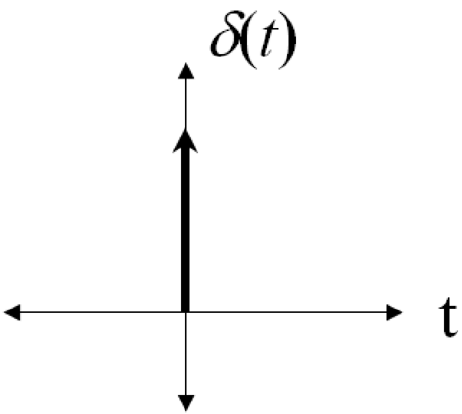
$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$



- Unit Impulse:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \ne 0 \end{cases}$$

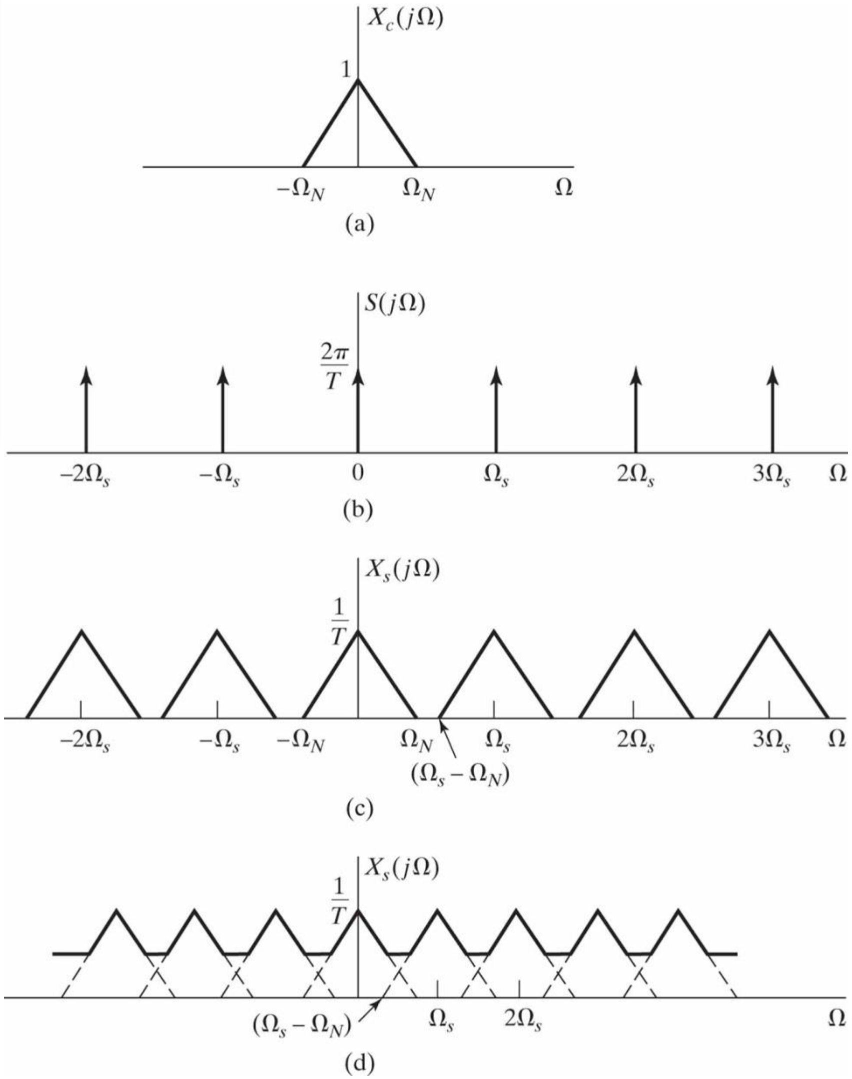
$$\int_{-a}^{b} \delta(t)dt = 1 \quad \text{for any } a > 0 \text{ and } b > 0.$$



Frequency-Domain Representation of Sampling

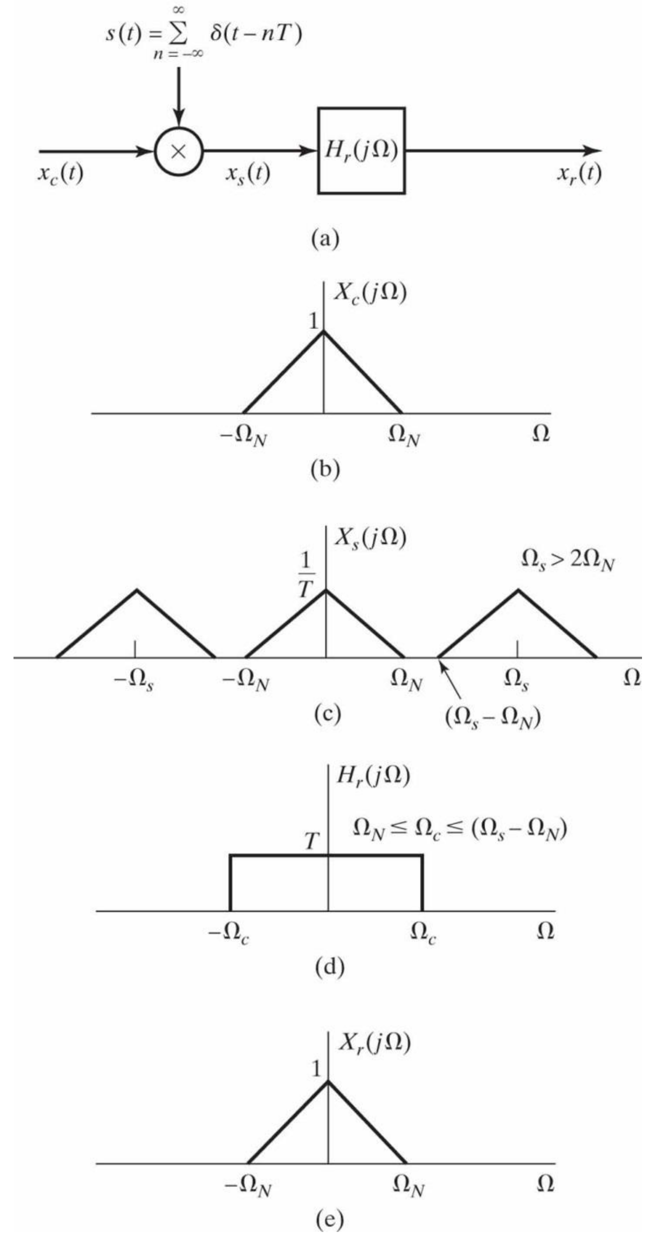
- $S(j\Omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega - k\Omega_s)$
- $X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega)$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\Omega - k\Omega_s))$$



Recovery of $x_c(t)$ from $x_s(t)$

- If you can recover $x_c(t)$ from $x_s(t)$, you can recover $x_c(t)$ from $x[n]$.
- Recovery is possible through an ideal low-pass filter when $\Omega_s > 2\Omega_N$.



Nyquist-Shannon Sampling Theorem

Let $x_c(t)$ be a band-limited signal with

$$X_c(j\Omega) = 0 \text{ for } |\Omega| \geq \Omega_N.$$

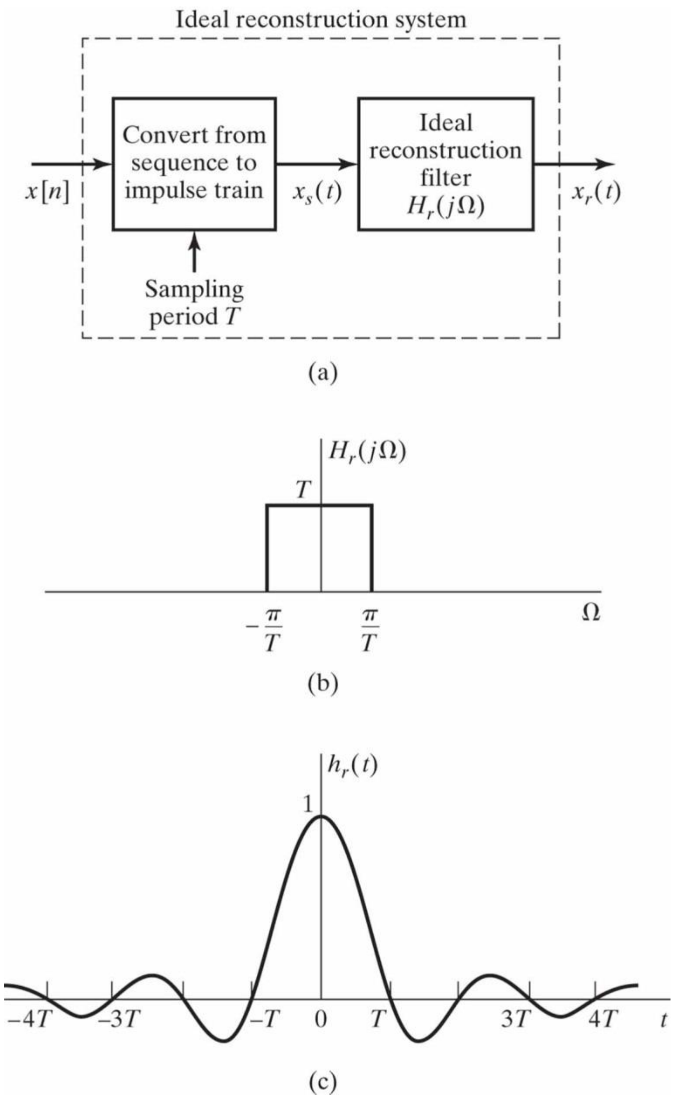
Then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT),\ -\infty < n < \infty$, if

$$\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N.$$

- $2\Omega_N$ is called the Nyquist rate.
- Under certain conditions, a CT signal can be completely represented by and recoverable from samples
- A low-pass signal can be reconstructed from samples, if the sampling rate is high enough. Because it is a low-pass signal, the change between two close samples is constrained (or expected).

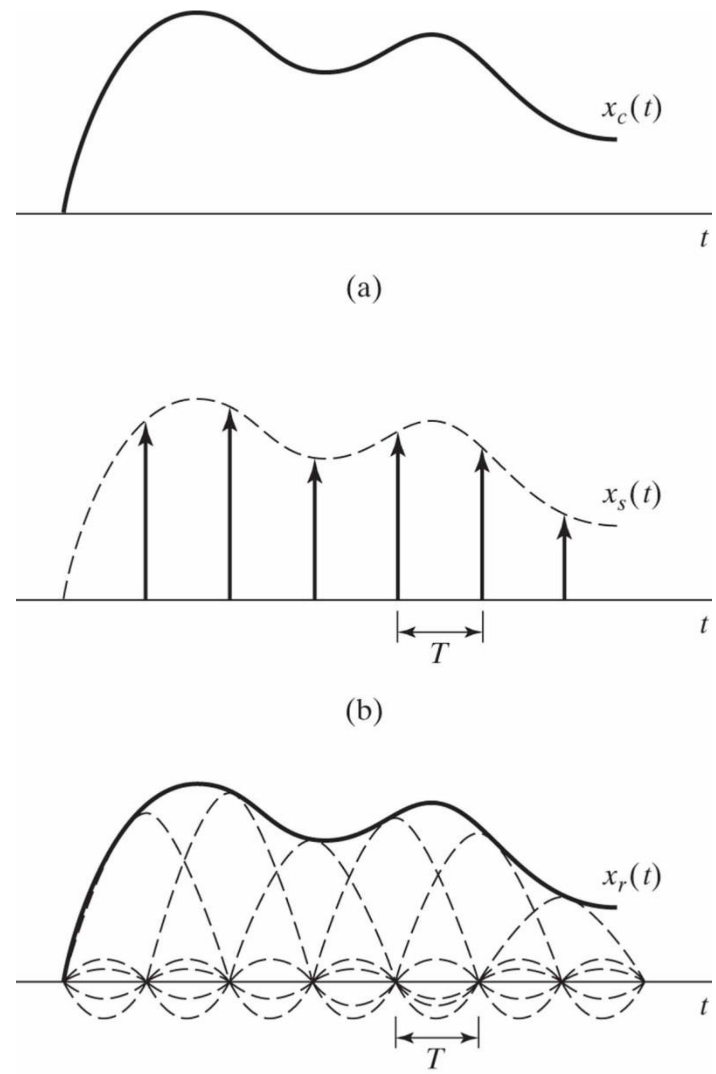
Recovery of $x_c(t)$ from $x_s(t)$

- $h_r(t) = \frac{\sin(\frac{\pi t}{T})}{\frac{\pi t}{T}}$
- $x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$
- $x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\frac{\pi(t-nT)}{T})}{\frac{\pi(t-nT)}{T}}$



Recovery of $x_c(t)$ from $x[n]$

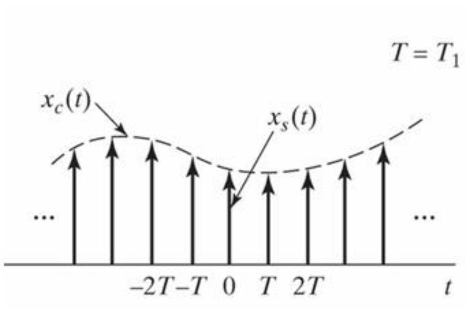
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\frac{\pi(t-nT)}{T})}{\frac{\pi(t-nT)}{T}}$$



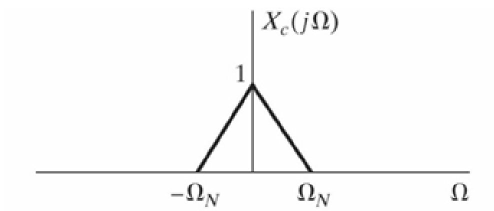
(a)
(b)
(c)

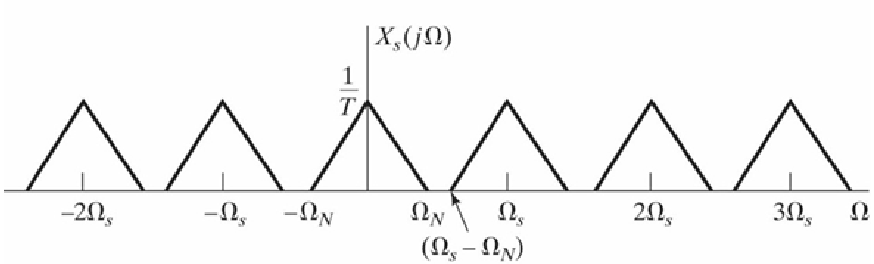
Summary

- $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, $x_s(t) = x_c(t)s(t)$

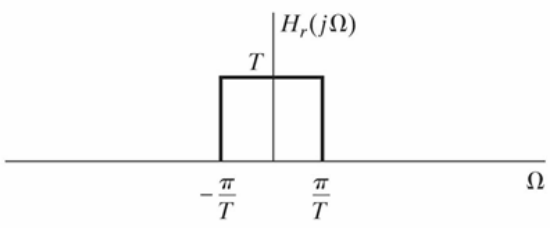


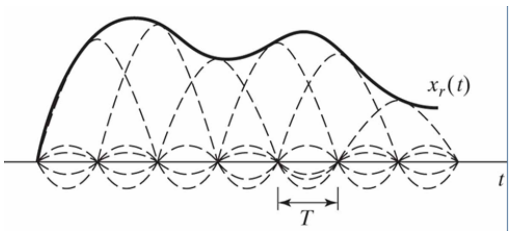
- $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$





- $X_r(j\Omega) = X_s(j\Omega)H_r(j\Omega)$, $x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\frac{\pi(t-nT)}{T})}{\frac{\pi(t-nT)}{T}}$





Frequency-Domain Relationship between $x[n]$ and $x_s(t)$

- Relationship between $X(e^{j\omega})$ and $X_s(j\Omega)$

$$X(e^{j\omega}) = X_s\left(j\frac{\omega}{T}\right)$$

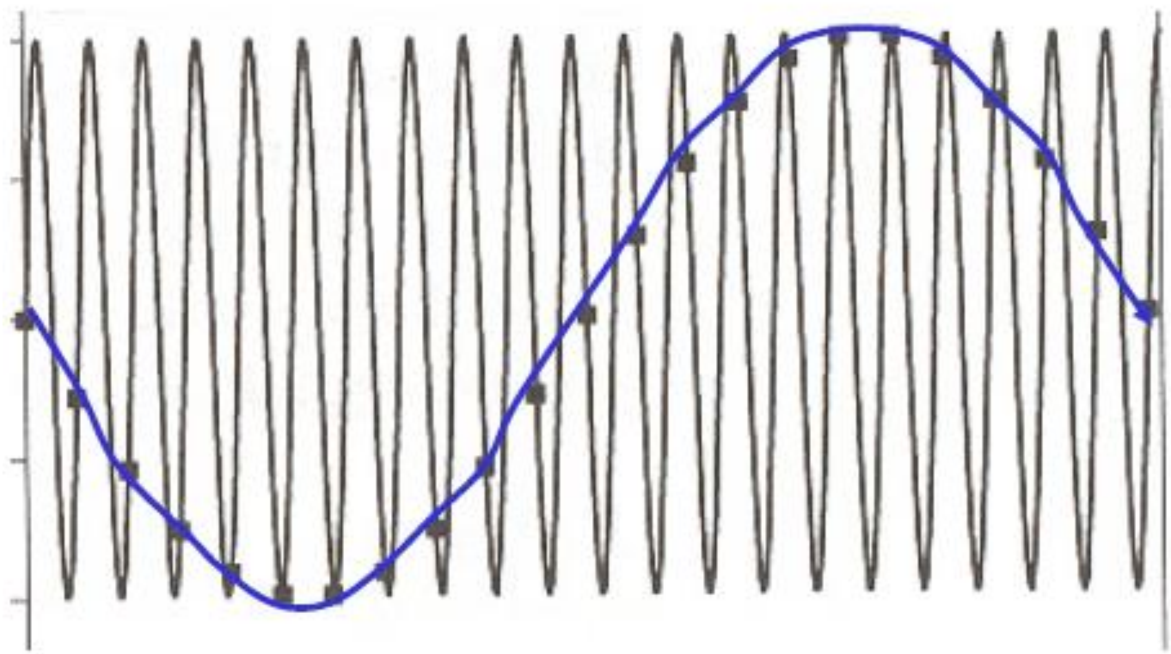
$$X_s(j\Omega) = X(e^{j\Omega T})$$

- Recall that $X(e^{j\omega})$ is always periodic

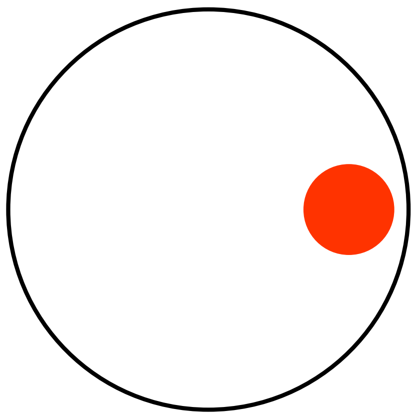
Aliasing

Undersampling Causes Aliasing

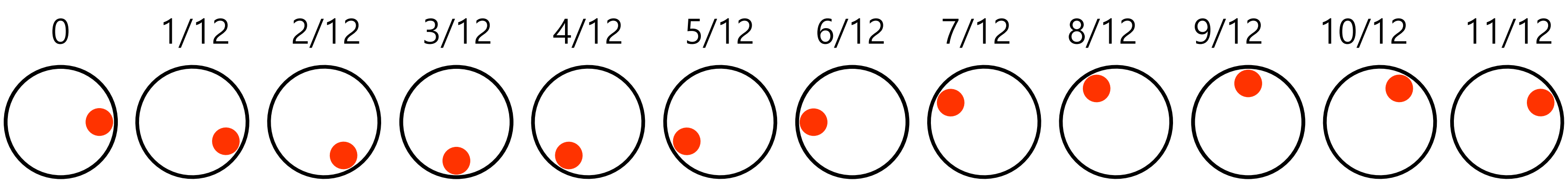
- Undersampling: sampling rate is less than Nyquist rate



Undersampling Causes Aliasing

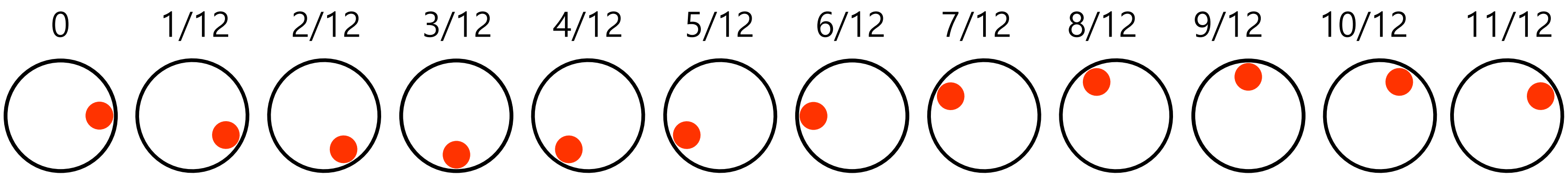


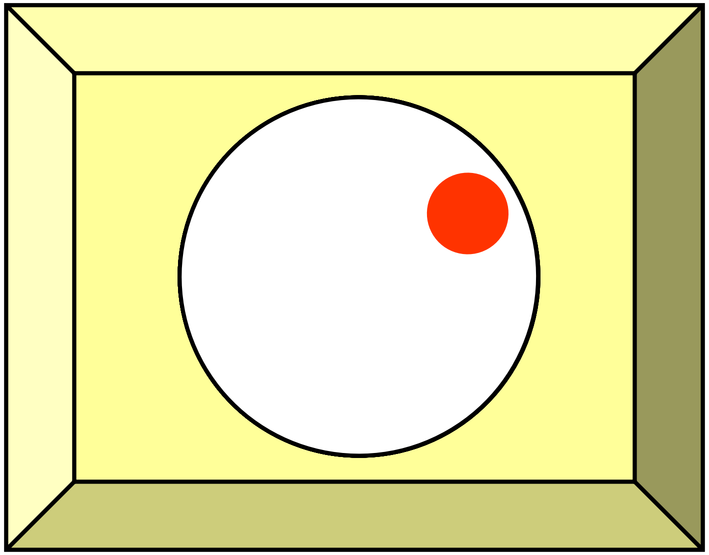
- Rotating disk
  - 1 rotation/second
- To avoid aliasing, it should be motion-pictured with at least 2 frames/s.



Undersampling Causes Aliasing

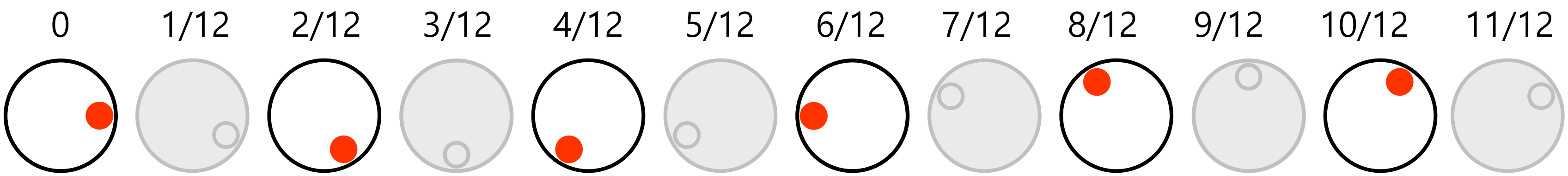
- 12 frames/s

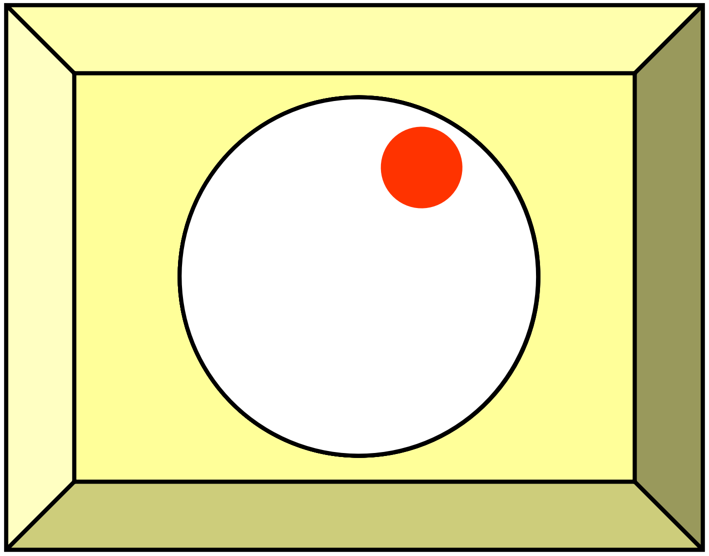




Undersampling Causes Aliasing

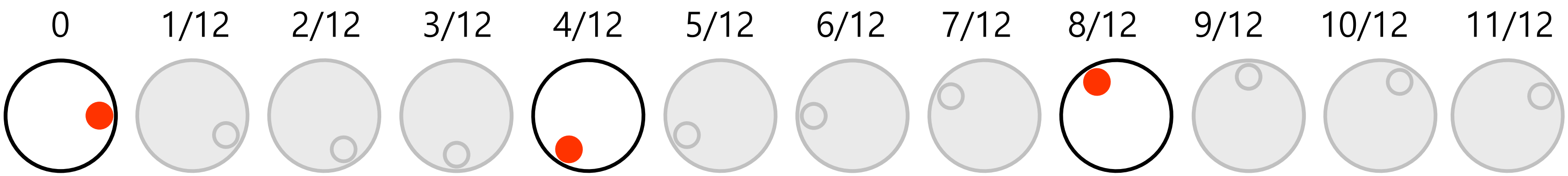
- 6 frames/s

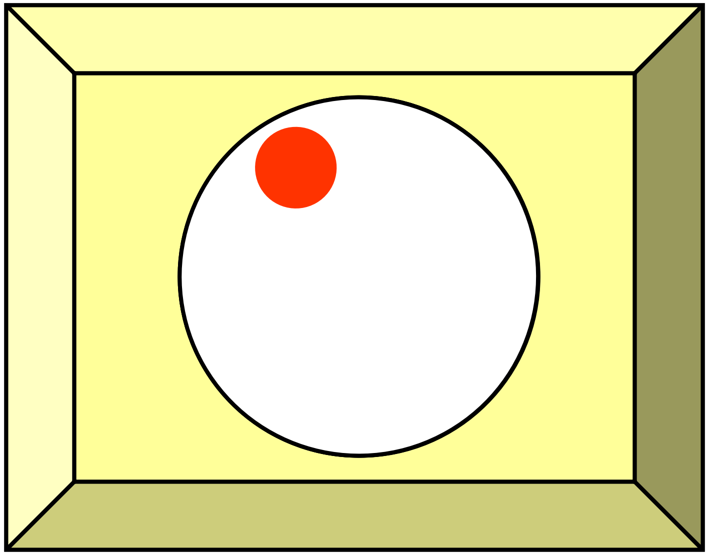




Undersampling Causes Aliasing

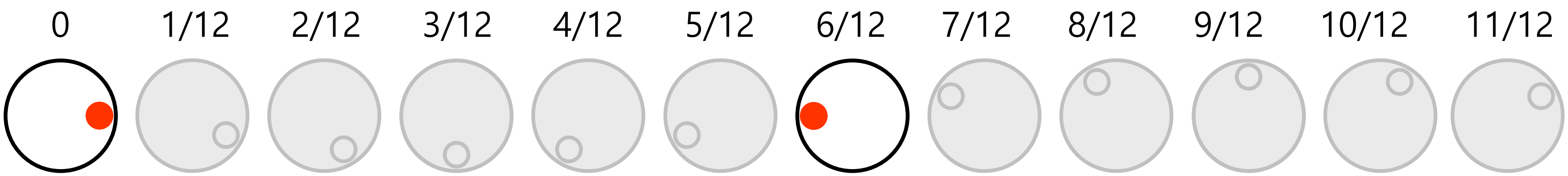
- 3 frames/s

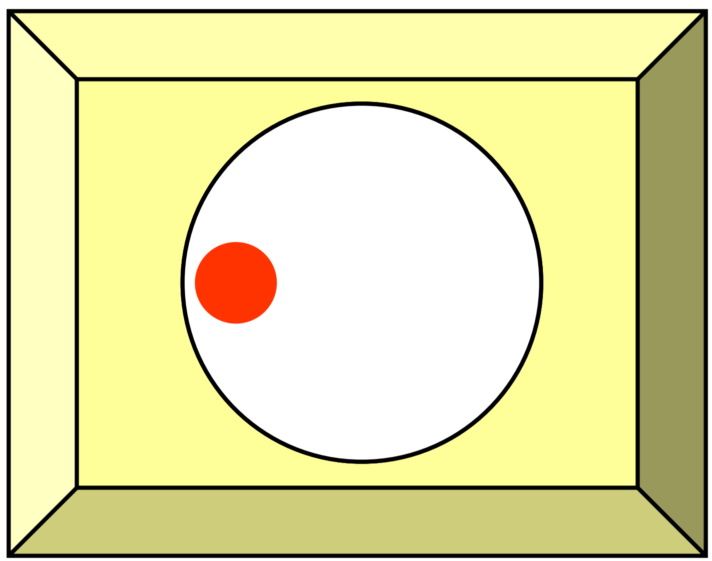




Undersampling Causes Aliasing

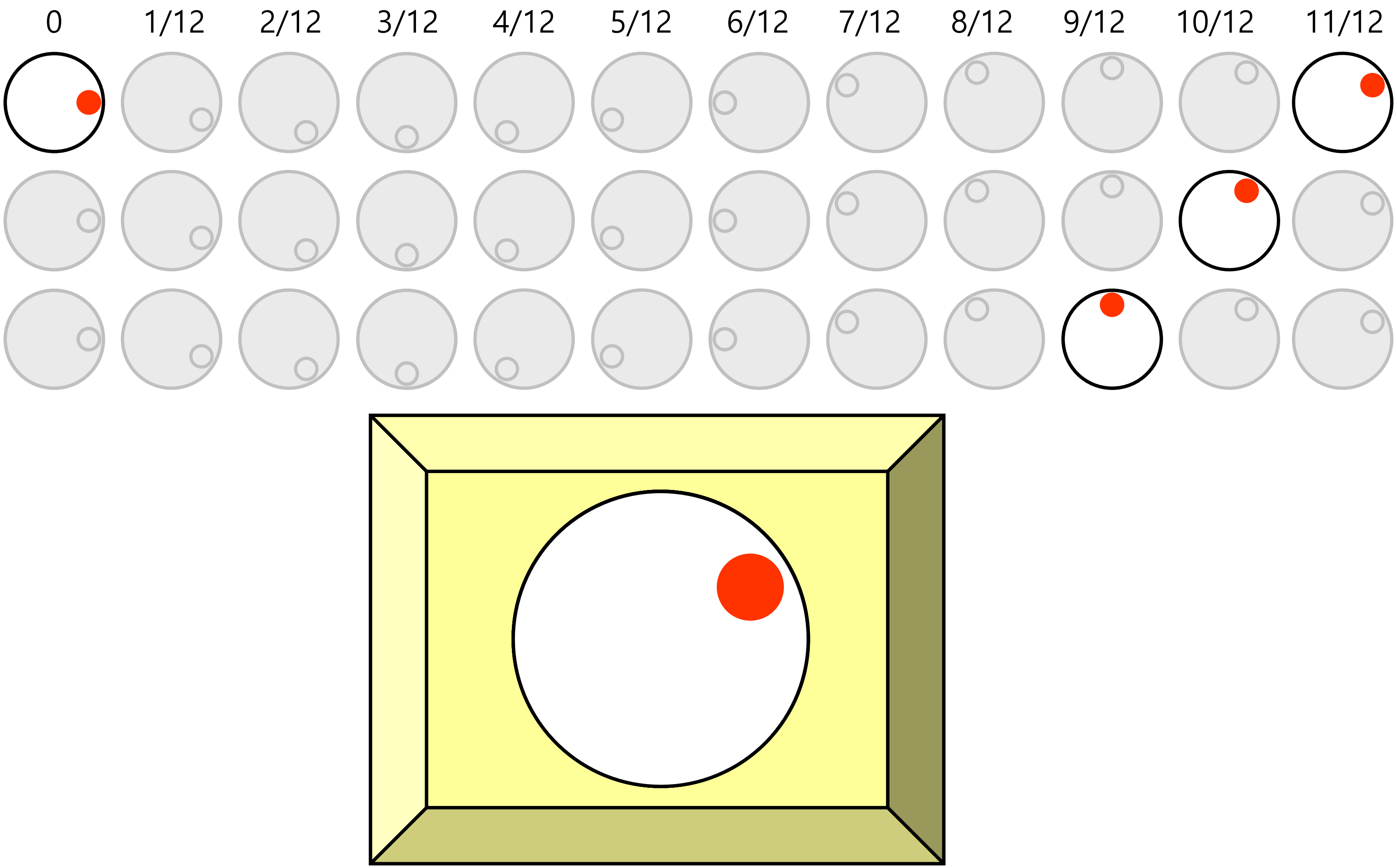
- 2 frames/s





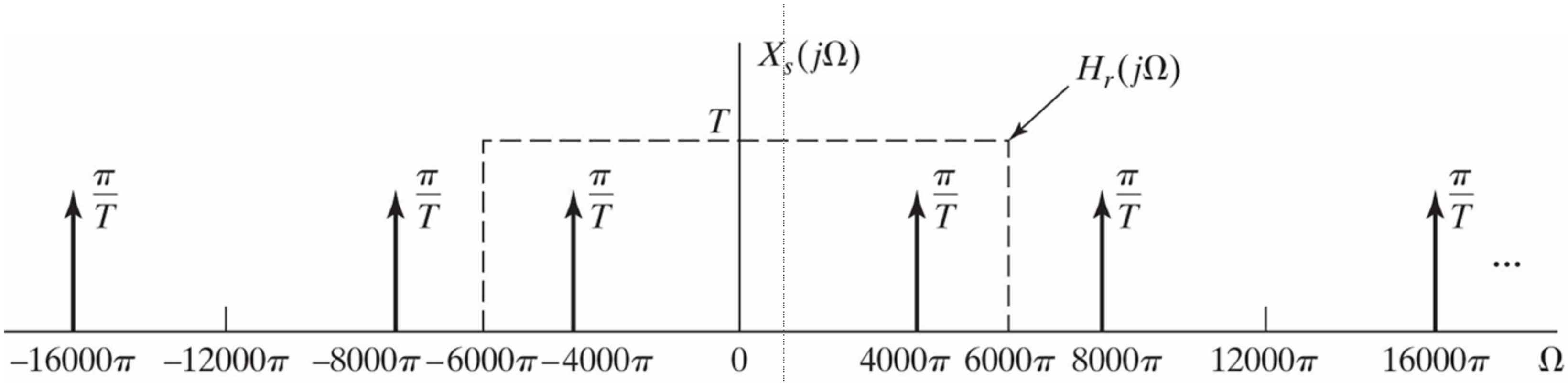
Undersampling Causes Aliasing

- 12/11 = 1.09 frames/s



Examples

- $x_c(t) = \cos(4000\pi t), \;\; T = 1/6000.$

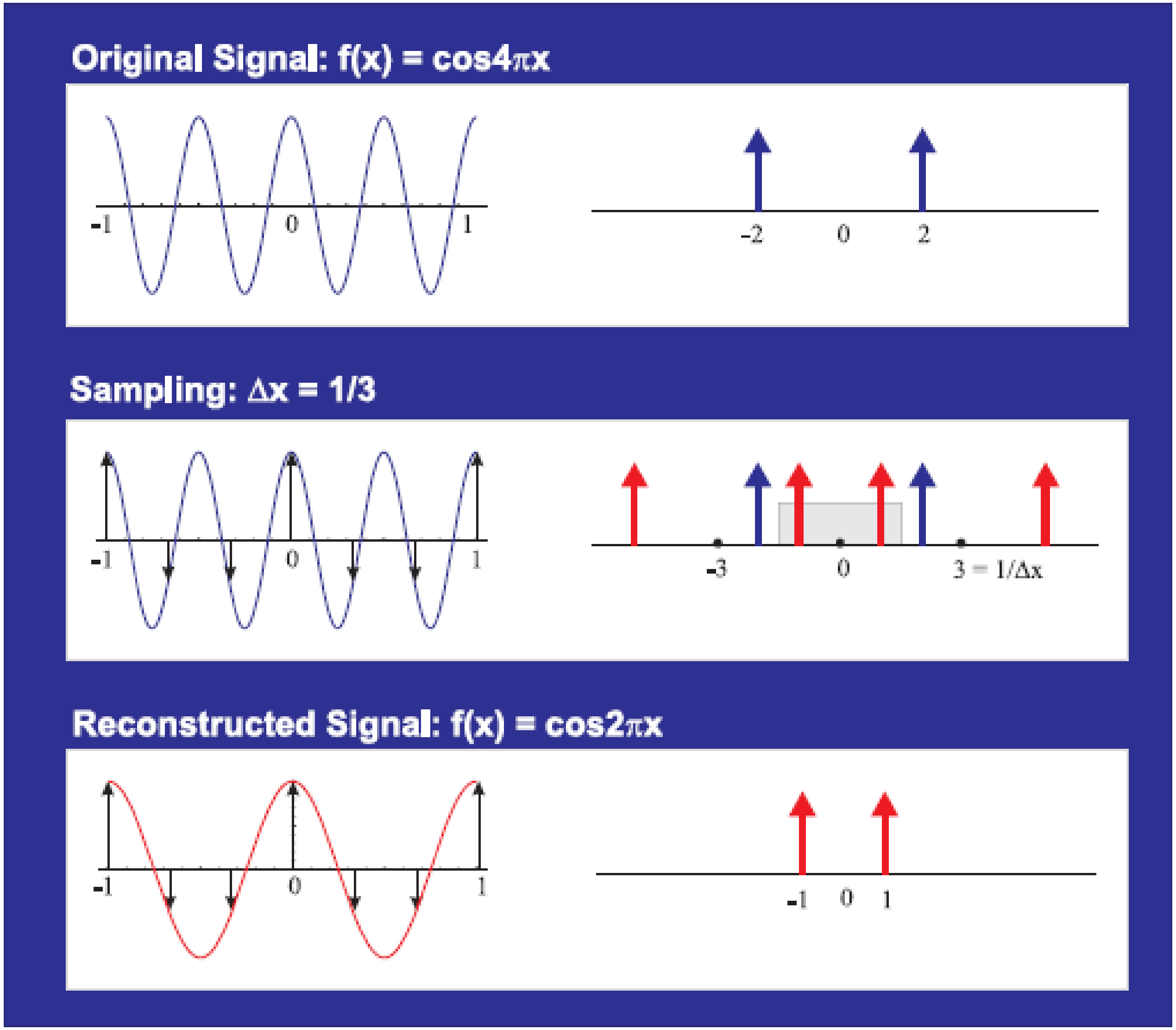


- $1 \leftrightarrow 2\pi\delta(\Omega)$
- $\cos(\Omega_0 t) \leftrightarrow \pi(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0))$
- $\sin(\Omega_0 t) \leftrightarrow \frac{\pi}{j}(\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0))$

Examples

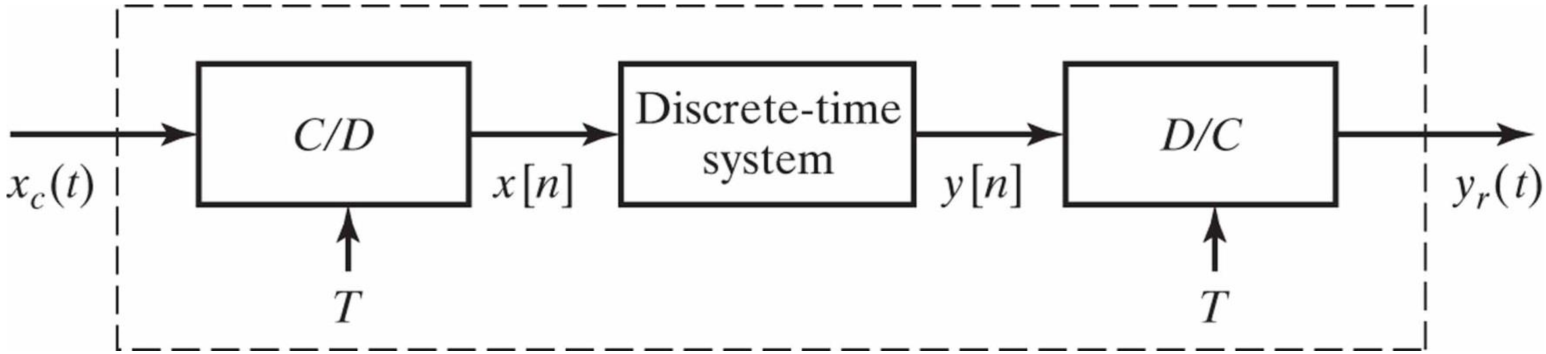
- $x_c(t) = \cos(16000\pi t), \;\; T = 1/6000.$

Examples


Original Signal: f(x) = cos4πx
-1
0
1
-2
0
2
Sampling: Δx = 1/3
-1
0
1
-3
0
3 = 1/Δx
Reconstructed Signal: f(x) = cos2πx
-1
0
1
-1
0
1

DT Processing of CT Signals

C/D and D/C conversions



- C/D

$$X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

- D/C

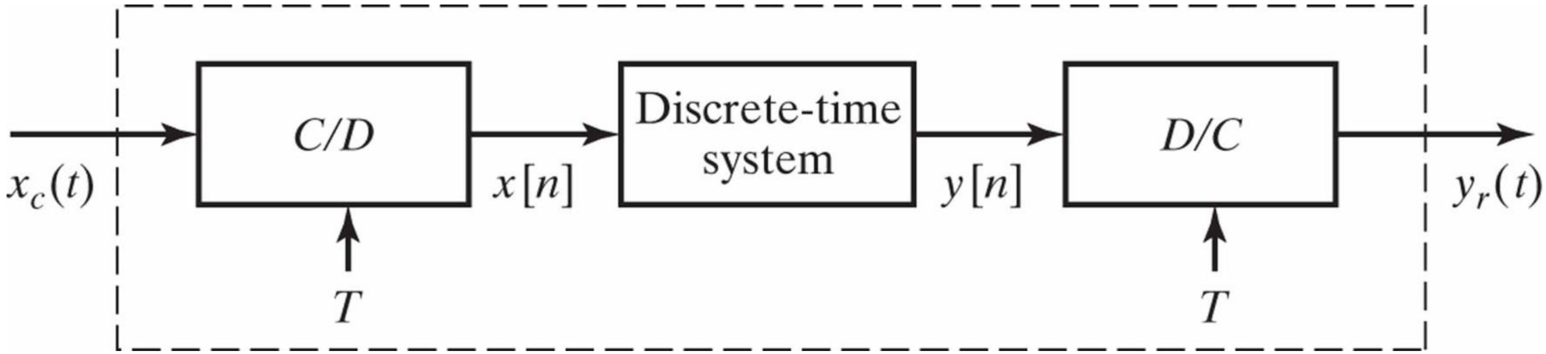
$$Y_r(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T}) = \begin{cases} TY(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & \text{Otherwise} \end{cases}$$

Relationship between $x[n]$ and $x_s(t)$

$$X(e^{j\omega}) = X_s\left(j\frac{\omega}{T}\right),$$

$$X_s(j\Omega) = X(e^{j\Omega T})$$

Overall System



- Effective Frequency Response

$$H_{\text{eff}}(j\Omega) = \begin{cases} H\left(e^{j\Omega T}\right), & |\Omega| < \dfrac{\pi}{T} \\ 0, & \text{Otherwise} \end{cases}$$

- Assumptions
  - $x_c(t)$ is band-limited
  - $\frac{2\pi}{T}$ satisfies the Nyquist rate

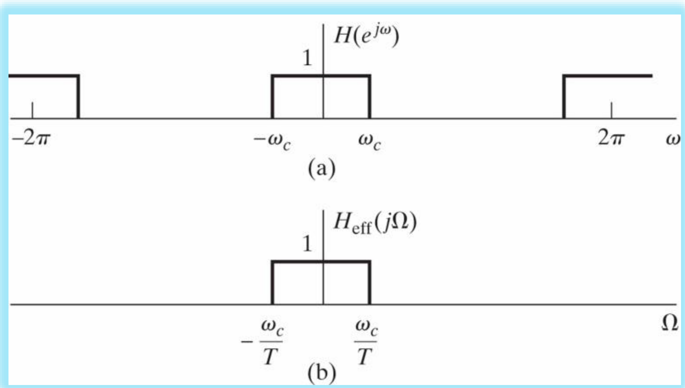
Relationship between $x[n]$ and $x_s(t)$

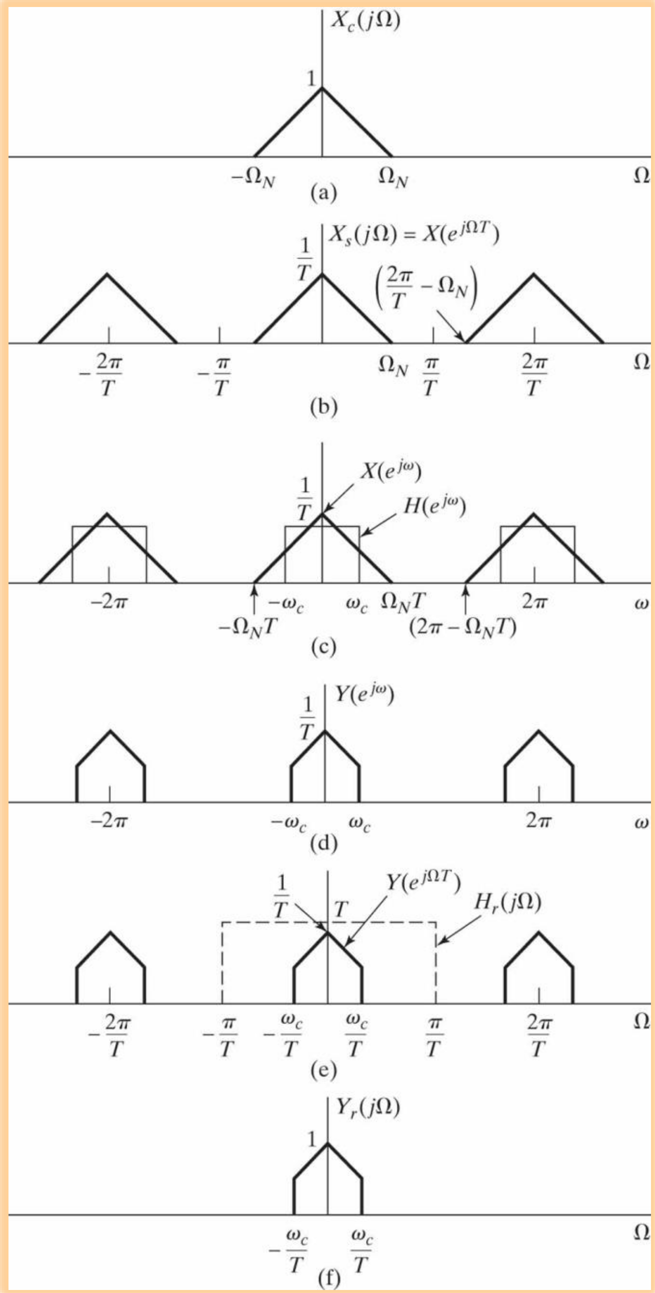
$$X\left(e^{j\omega}\right) = X_s\left(j\frac{\omega}{T}\right),$$

$$X_s(j\Omega) = X\left(e^{j\Omega T}\right)$$

Example 1

- $H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$

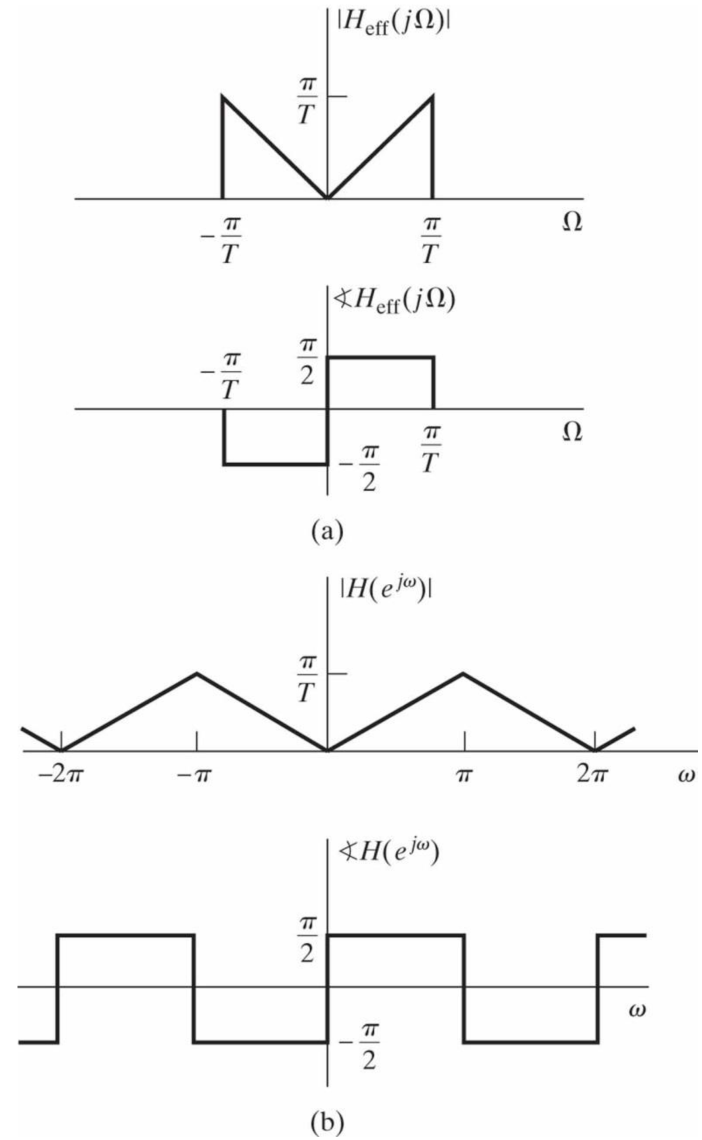




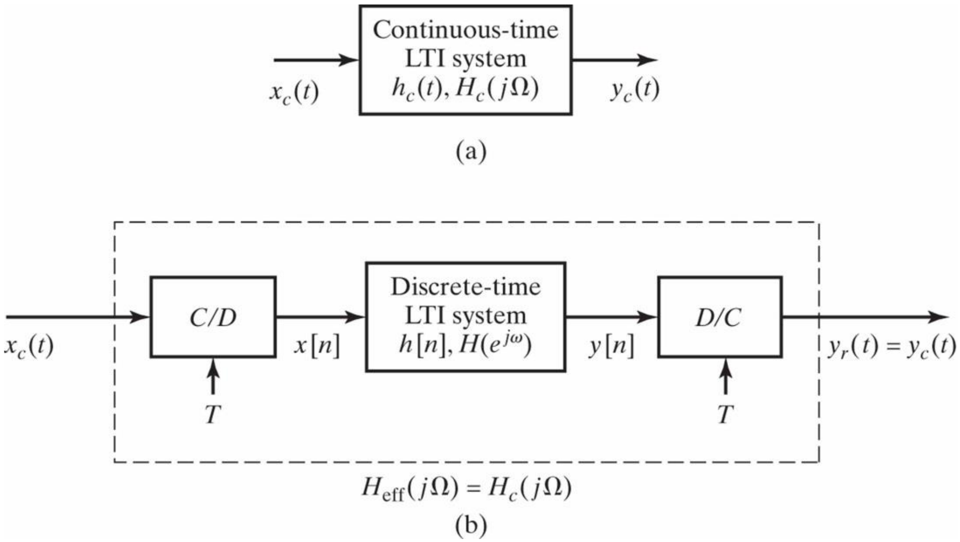
Example 2

- $y_c(t) = \frac{d}{dt} x_c(t)$

$$\Rightarrow\ h[n] = \begin{cases} 0, & n = 0 \\ \dfrac{(-1)^n}{nT}, & n \neq 0 \end{cases}$$



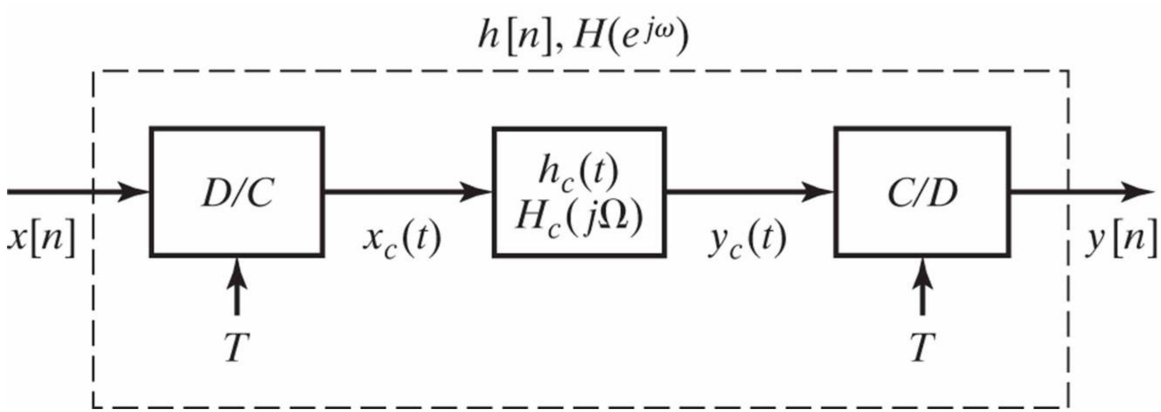
Impulse Invariance



- $h[n] = Th_c(nT)$
- Example
  - Ideal lowpass filter $h[n]$ with cutoff frequency ω_c

CT Processing of DT Signals

CT Processing of DT Signals



- It is rarely used, but provides a useful interpretation of some DT systems
- Main results

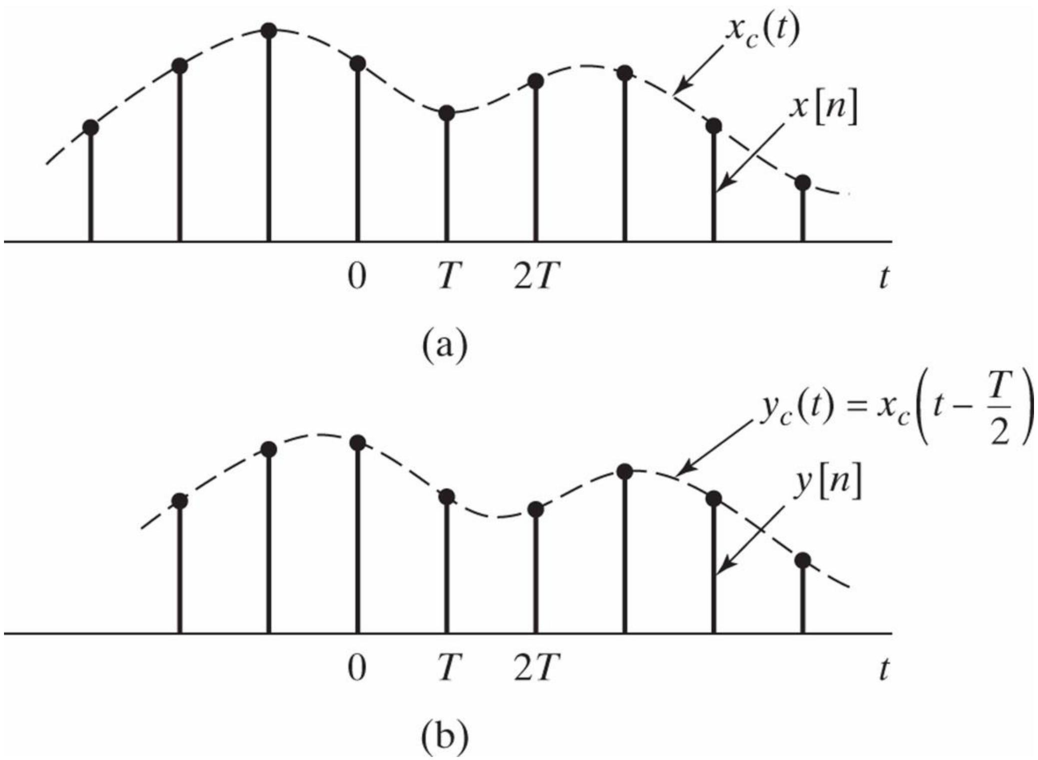
$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T}\right), \qquad |\omega| < \pi$$

$$H_c(j\Omega) = H(e^{j\Omega T}), \qquad |\Omega| < \frac{\pi}{T}.$$

Example – Fractional Delay

- $H(e^{j\omega}) = e^{-j\omega\Delta}, \ |\omega| < \pi$

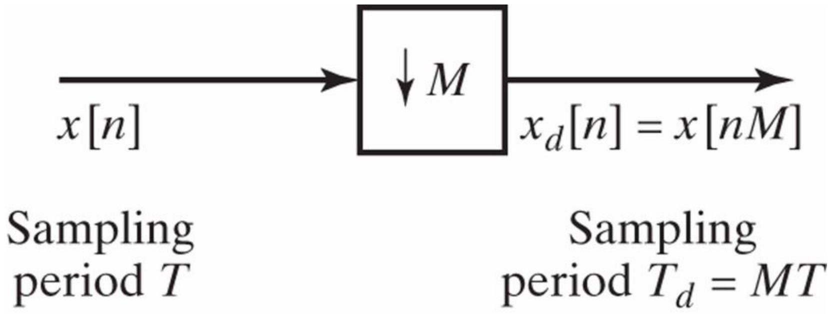
$$\Longrightarrow \ h[n] = \frac{\sin \pi(n - \Delta)}{\pi(n - \Delta)}$$



Changing Sampling Rate

Reducing Sampling Rate by an Integer Factor M

sampling rate compressor

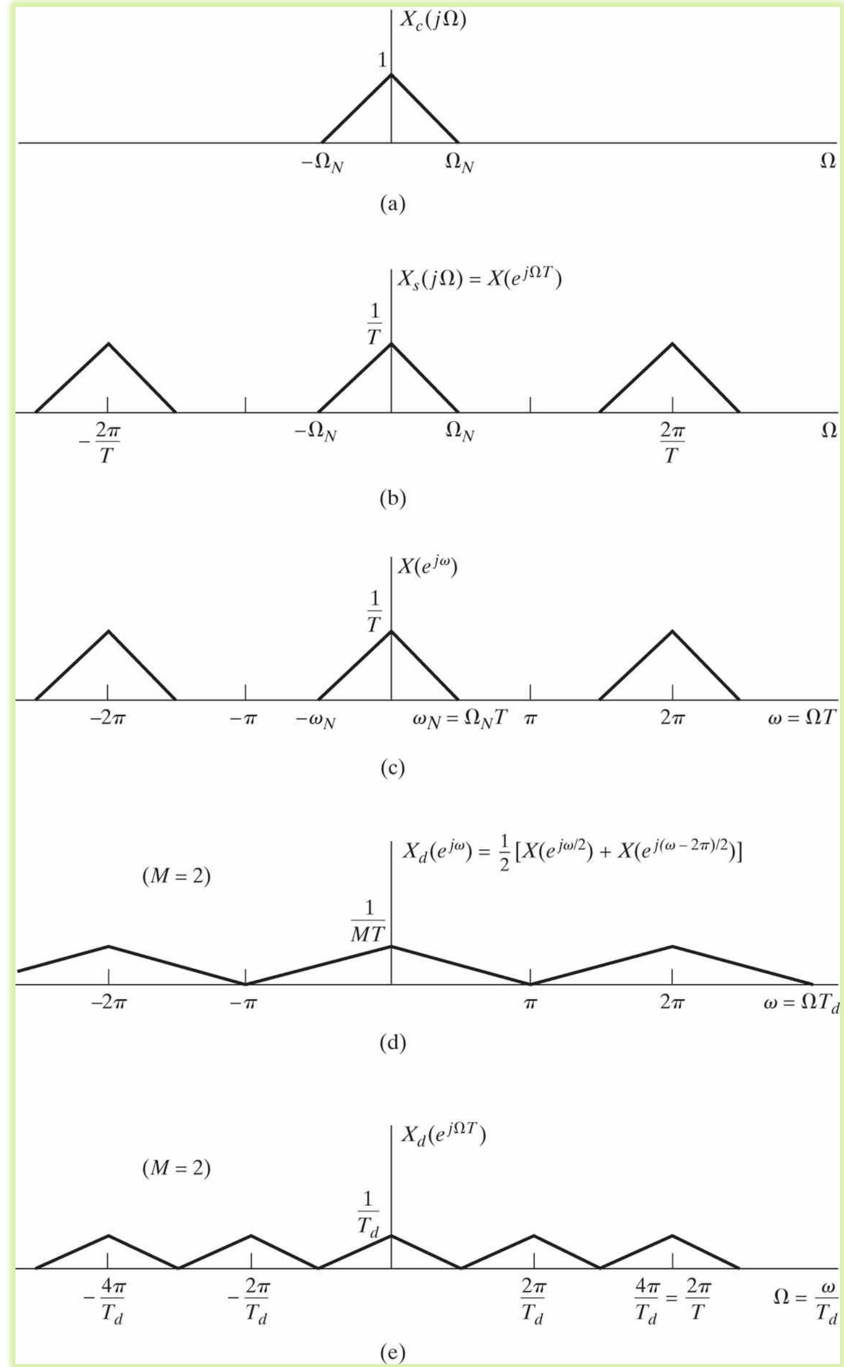


- Time domain

$$x_d[n] = x[nM]$$

- Frequency domain

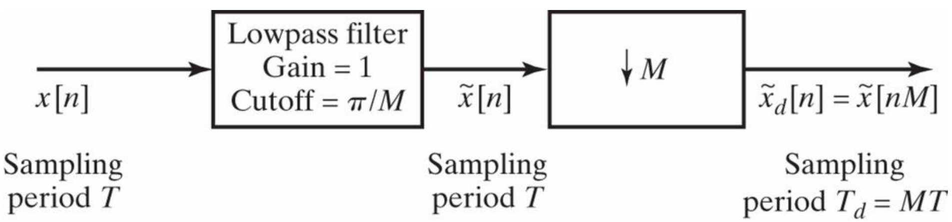
$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega}{M}-\frac{2\pi k}{M}\right)}\right)$$

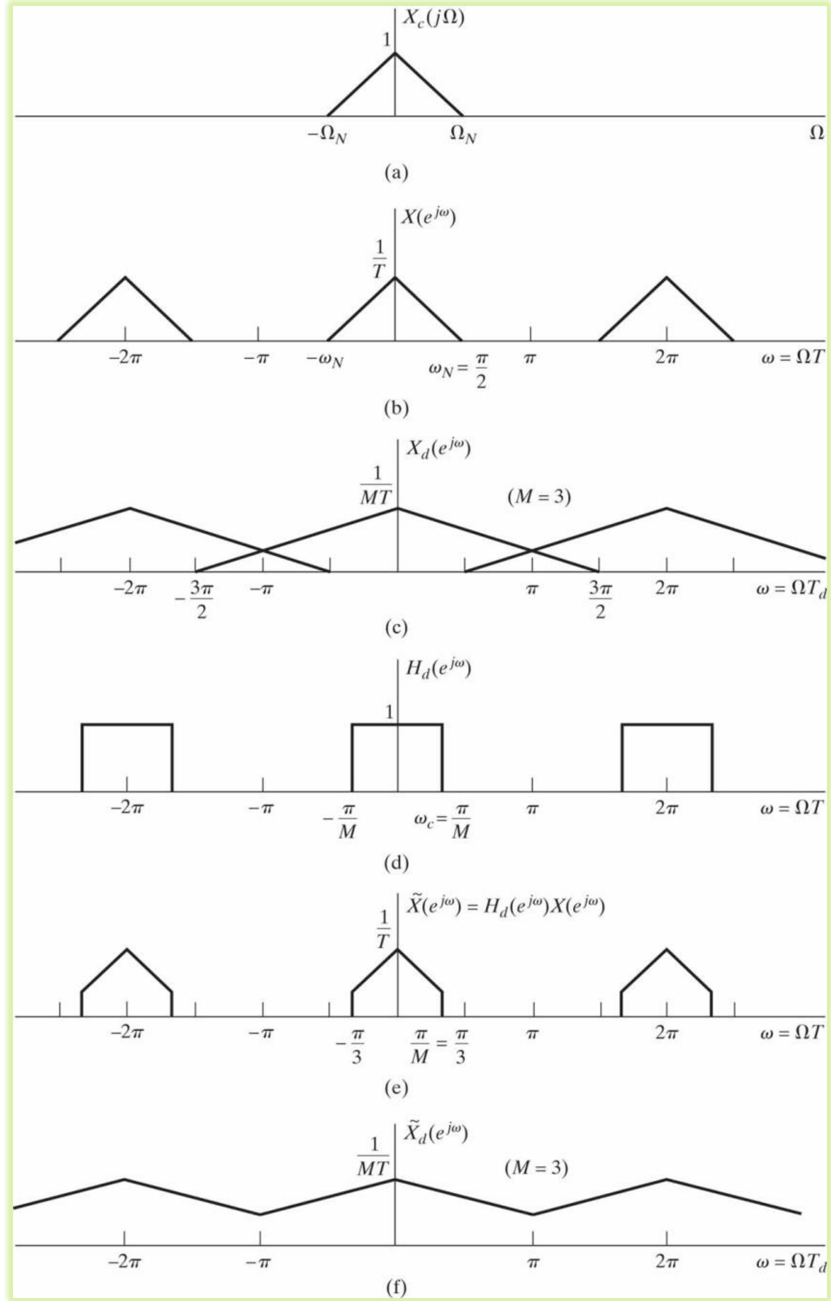


Reducing Sampling Rate by an Integer Factor M

- To avoid aliasing, we need
  - $X(e^{j\omega}) = 0$ if $\omega_N < |\omega| < \pi$
  - $\omega_N < \frac{\pi}{M}$

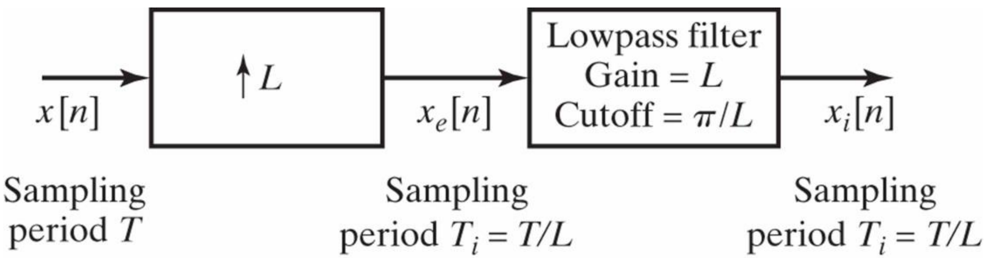
- Anti-aliasing filter can be used





Increasing Sampling Rate by an Integer Factor L

sampling rate expander



- Input and output

$$x[n] = x_c(nT)$$

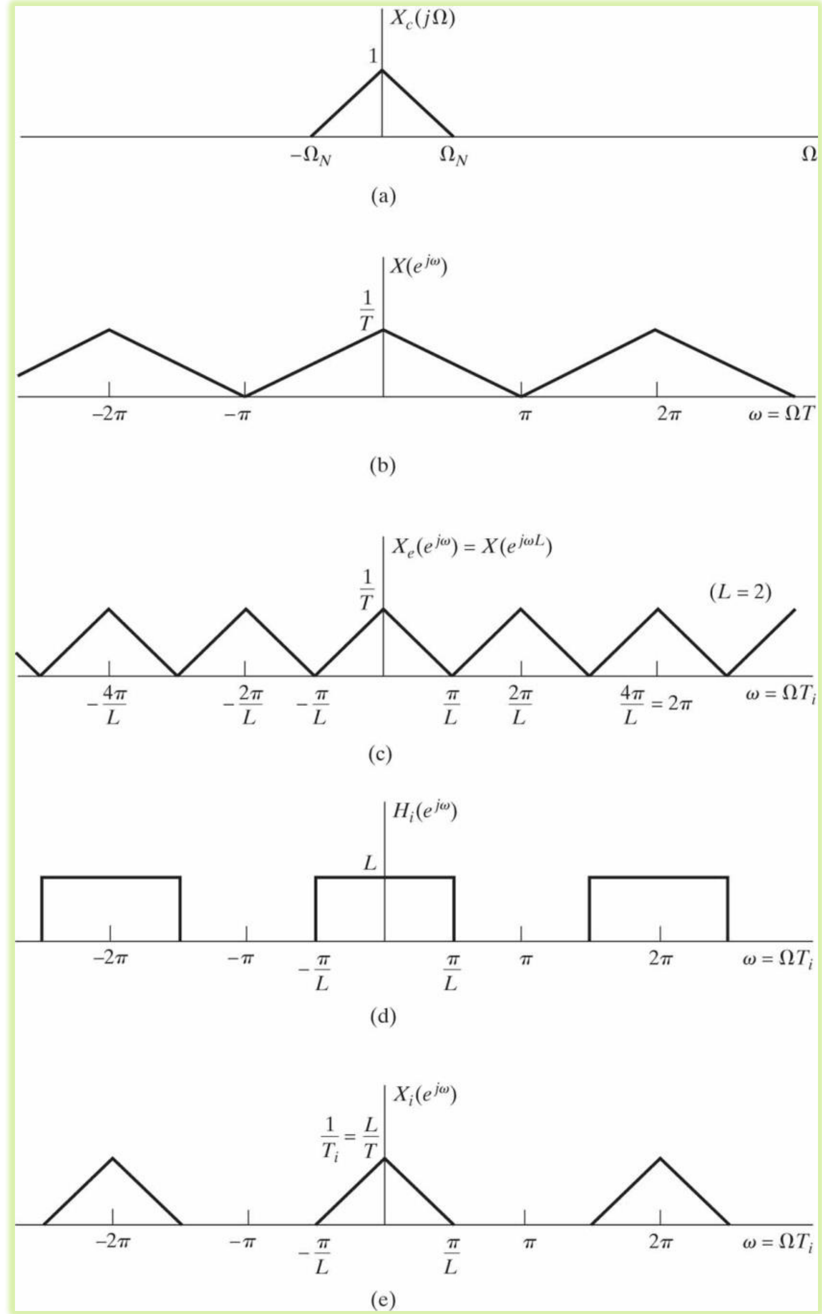
$$x_i[n] = x_c\left(n\frac{T}{L}\right)$$

- Intermediate signal

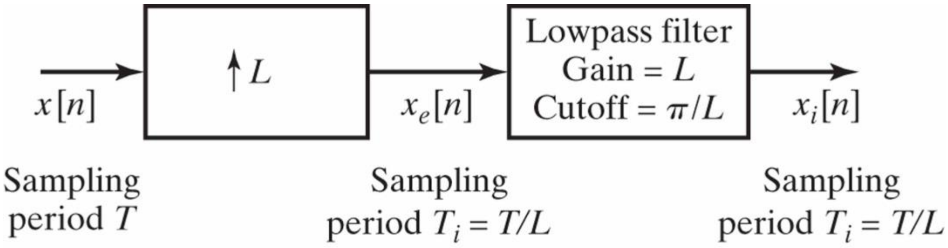
$$x_e[n] = \begin{cases} x[\frac{n}{L}] & \text{if } n \text{ is a multiple of } L \\ 0 & \text{otherwise} \end{cases}$$

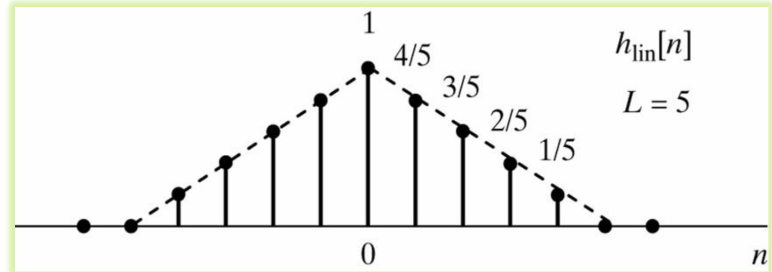
- Output in terms of input

$$x_i[n] = \sum_{k=-\infty}^{\infty} x[k] \frac{\sin \frac{\pi(n-kL)}{L}}{\frac{\pi(n-kL)}{L}}$$



Ideal and Linear Interpolation Filters



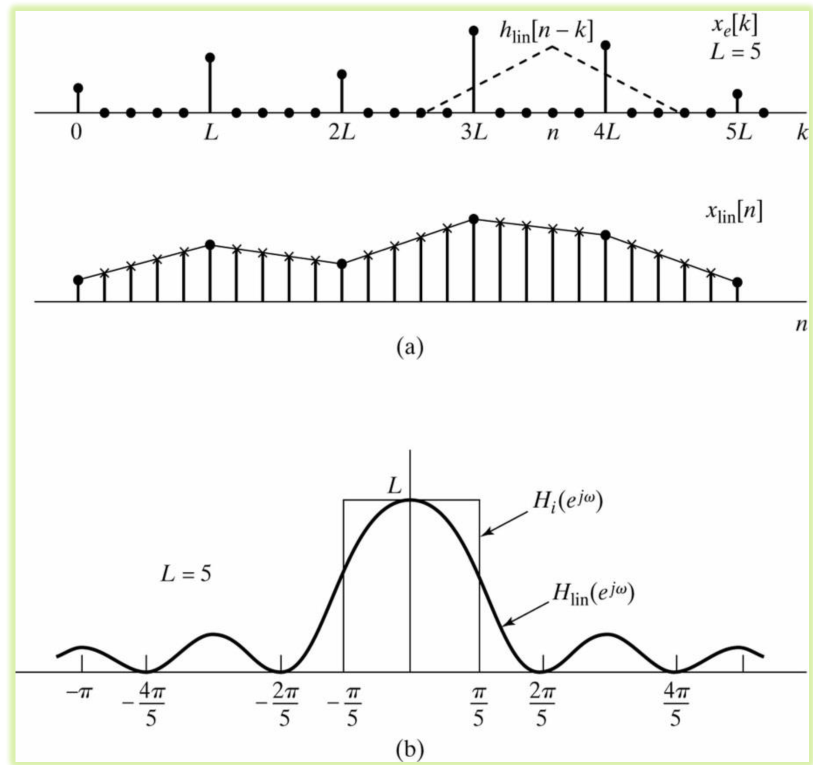


- $x_e[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-kL]$
- $x_i[n] = x_e[n] * h_i[n] = \sum_{k=-\infty}^{\infty} x[k]h_i[n-kL]$
- Ideal filter

$$h_i[n] = \frac{\sin \frac{\pi n}{L}}{\frac{\pi n}{L}}$$

- Linear filter

$$h_{\mathrm{lin}}[n] = \begin{cases} 1 - \frac{|n|}{L}, & -L \le n \le L \\ 0, & \text{otherwise} \end{cases}$$



Changing Sampling Rate by a Noninteger Factor