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# Other Transforms

## *(IEG4160 – Image and Video Processing)*

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# Matrix Representation of Transform

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Transform

$$v = Au$$

- $u$ :  $N \times 1$  input vector
- $v$ :  $N \times 1$  output vector
- $A$ :  $N \times N$  matrix (transform)

Inverse transform

$$u = A^{-1}v$$

# Unitary Transform

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- $A^{-1} = A^H$

- $v = Au$

- $u = A^H v$

- $A = \begin{bmatrix} b_0^H \\ b_1^H \\ \vdots \\ b_{N-1}^H \end{bmatrix}, \quad A^H = \begin{bmatrix} b_0 & b_1 & \cdots & b_{N-1} \end{bmatrix}$

- $b_i$ 's: basis vectors

- $AA^H = I$

- $b_i^H \cdot b_j = \delta(i - j)$

# Energy Conservation of Unitary Transform

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$$\sum_{k=0}^{N-1} |v(k)|^2 = \|v\|^2 = \|u\|^2 = \sum_{n=0}^{N-1} |u(n)|^2$$

- Unitary transform does not change the length of input vector
- It is just an axis rotation

# Matrix Representation of DFT

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$$v(k) = \frac{1}{N} \sum_{n=0}^{N-1} u(n) W_N^{kn}$$

$$u(n) = \sum_{k=0}^{N-1} v(k) W_N^{-kn}$$

$$v = \frac{1}{N} \begin{bmatrix} W_N^{0 \cdot 0} & W_N^{0 \cdot 1} & \dots & W_N^{0 \cdot (N-1)} \\ W_N^{1 \cdot 0} & W_N^{1 \cdot 1} & \dots & W_N^{1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 0} & W_N^{(N-1) \cdot 1} & \dots & W_N^{(N-1) \cdot (N-1)} \end{bmatrix} u$$

$$u = \begin{bmatrix} W_N^{-0 \cdot 0} & W_N^{-0 \cdot 1} & \dots & W_N^{-0 \cdot (N-1)} \\ W_N^{-1 \cdot 0} & W_N^{-1 \cdot 1} & \dots & W_N^{-1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{-(N-1) \cdot 0} & W_N^{-(N-1) \cdot 1} & \dots & W_N^{-(N-1) \cdot (N-1)} \end{bmatrix} v$$

# Unitary DFT

$$\bar{v} = Au = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{0 \cdot 0} & W_N^{0 \cdot 1} & \dots & W_N^{0 \cdot (N-1)} \\ W_N^{1 \cdot 0} & W_N^{1 \cdot 1} & \dots & W_N^{1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1) \cdot 0} & W_N^{(N-1) \cdot 1} & \dots & W_N^{(N-1) \cdot (N-1)} \end{bmatrix} u$$

$$u = A^H \bar{v} = \frac{1}{\sqrt{N}} \begin{bmatrix} W_N^{-0 \cdot 0} & W_N^{-0 \cdot 1} & \dots & W_N^{-0 \cdot (N-1)} \\ W_N^{-1 \cdot 0} & W_N^{-1 \cdot 1} & \dots & W_N^{-1 \cdot (N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{-(N-1) \cdot 0} & W_N^{-(N-1) \cdot 1} & \dots & W_N^{-(N-1) \cdot (N-1)} \end{bmatrix} \bar{v}$$

$$\bar{v}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} u(n) W_N^{kn}$$

$$u(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \bar{v}(k) W_N^{-kn}$$

# Karhunen Loeve Transform (KLT)

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- $u$ :  $N \times 1$  random vector (real)
- $R = E[uu^T]$ : autocorrelation matrix of  $u$ 
  - $R^T = R$
  - $a^T R a \geq 0$  for all  $a$
  - $R$  is a symmetric non-negative definite matrix
- Eigenvalue decomposition (results of linear algebra)

$$R\phi_k = \lambda_k \phi_k, \quad 0 \leq k < N$$

- Orthonormal set of eigenvectors:  $\phi_i^T \phi_j = \delta(i - j)$
- Sorting of eigenvalues:  $\lambda_0 \geq \lambda_1 \geq \dots \lambda_{N-1} \geq 0$
- $\Phi = [\phi_1, \phi_2, \dots, \phi_{N-1}]$ 
  - $\Phi^{-1} = \Phi^T$ :  $\Phi$  is orthogonal (real and unitary)
  - $R\Phi = \Phi\Lambda$  where  $\Lambda = \text{Diag}\{\lambda_k\}$

# KLT

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- KLT

- $v = \Phi^T u$

- $u = \Phi v$

- KLT is optimum in many senses

- KLT coefficients are decorrelated

- KLT minimizes the basis restriction mean square error

$$u \Rightarrow A \Rightarrow I_m \Rightarrow B \Rightarrow z$$

- $J_m = \frac{1}{N} E \left[ \sum_{n=0}^{N-1} (u(n) - z(n))^2 \right]$  is minimized when  
 $A = \Phi^T, B = A^{-1} = \Phi$

- For each  $m$ , among all orthogonal transforms, KLT packs the maximum energy in the first  $m$  samples

- KLT depends on input characteristics

- Fast KLT does not exist in general



# Discrete Cosine Transform (DCT)

■  $v = Cu, \quad u = C^{-1}v$

■  $C(k, n) = \begin{cases} \frac{1}{\sqrt{N}} & k = 0 \\ \sqrt{\frac{2}{N}} \cos \frac{\pi(2n+1)k}{2N} & \text{otherwise} \end{cases}$

■ Properties

- DCT is orthogonal, i.e.  $C^{-1} = C^T$
- DCT is not the real part of DFT
- There exist fast DCTs:  $O(N \log N)$
- DCT is very close to the optimum KLT, when the input has correlation matrix

$$R = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix}$$

and  $\rho$  is close to 1

- DCT is widely used for image compression

# Discrete Sine Transform (DST)

■  $v = Su, \quad u = S^{-1}v$

■  $S(k, n) = \sqrt{\frac{2}{N+1}} \sin \frac{\pi(k+1)(n+1)}{N+1}$

■ Properties

■ DST is orthogonal, i.e.  $S^{-1} = S^T$

■ The forward and inverse transforms are identical, i.e.  $S^{-1} = S$

■ DST is not the imaginary part of DFT

■ There exist fast DSTs:  $O(N \log N)$

■ DST is very close to the optimum KLT, when the input has correlation matrix

$$R = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^{N-1} \\ \rho & 1 & \rho & \dots & \rho^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \dots & 1 \end{bmatrix}$$

and  $\rho$  lies within the interval  $(-0.5, 0.5)$

# The Hadamard Transform

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- Recursive definition of the transform matrix

$$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

- For example,

$$H_2 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

- Properties

- $H^{-1} = H^T = H$
- Fast implementation is possible:  $O(N \log N)$
- Hadamard transform requires almost no multiplications
- Relatively good energy compaction for highly correlated images

# The Haar Transform

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- Wavelet decomposition using  $H_1$  as the low-pass and high-pass filters
- In fact, the Haar transform is the simplest wavelet transform
- For example, the four sample Haar transform is given by

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

- Properties
  - $A^{-1} = A^T$
  - The Haar transform is extremely fast:  $O(N)$
  - Poor energy compaction