Discrete Fourier Transform

(IEG4160 – Image and Video Processing)

KIM Chang-Su

cskim@ie.cuhk.edu.hk

Room 707, SHB

Tel: 2609-8452

Discrete Fourier Transform (DFT)

- M-point input signal

$$f(x) : 0 \leq x \leq M-1$$

- Forward transform

$$\begin{aligned} F(u) &= \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \qquad (W_M \triangleq e^{-j2\pi/M}) \\ &= \frac{1}{M}\sum_{x=0}^{M-1} f(x)W_M^{ux} \qquad \text{for each } 0 \leq u \leq M-1 \end{aligned}$$

  - Computational load of DFT: approximately M^2 additions and multiplications

- Inverse transform

$$\begin{aligned} f(x) &= \sum_{u=0}^{M-1} F(u)e^{j2\pi ux/M} \\ &= \sum_{u=0}^{M-1} F(u)W_M^{-ux} \qquad \text{for each } 0 \leq x \leq M-1 \end{aligned}$$

Terminology

- $F(u) = R(u) + jI(u) = |F(u)|e^{-j\Phi(u)}$
  - Magnitude: $|F(u)| = (R^2(u) + I^2(u))^{\frac{1}{2}}$
  - Phase angle: $\Phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$
  - Power spectrum: $|F(u)|^2 = R^2(u) + I^2(u)$
- Basis vectors: $f(x) = \sum_{u=0}^{M-1} F(u) W_M^{-ux}$

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = F(0)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + F(1)\begin{bmatrix} 1 \\ W_4^{-1} \\ W_4^{-2} \\ W_4^{-3} \end{bmatrix} + F(2)\begin{bmatrix} 1 \\ W_4^{-2} \\ W_4^{-4} \\ W_4^{-6} \end{bmatrix} + F(3)\begin{bmatrix} 1 \\ W_4^{-3} \\ W_4^{-6} \\ W_4^{-9} \end{bmatrix}$$

  - Describing 4-D space with four basis vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ W_4^{-1} \\ W_4^{-2} \\ W_4^{-3} \end{bmatrix}, \begin{bmatrix} 1 \\ W_4^{-2} \\ W_4^{-4} \\ W_4^{-6} \end{bmatrix}, \begin{bmatrix} 1 \\ W_4^{-3} \\ W_4^{-6} \\ W_4^{-9} \end{bmatrix}$$

DFT is a Lossless Representation

- $\{f(x)\} \stackrel{\mathcal{F}}{\Longrightarrow} \{F(u)\} \stackrel{\mathcal{F}^{-1}}{\Longrightarrow} \{g(x)\}$, then

$$g(x) = f(x)$$

Properties of DFT

P1) The extensions are periodic

- The extension of f is periodic with period M

$$f(x + kM) = f(x)$$

- Similarly, the extension of F is periodic with period M

$$F(u + kM) = F(u)$$

Properties of DFT

P2) The DFT of a real sequence is conjugate symmetric

- Conjugate symmetric with respect to zero: $F(u) = F^*(-u)$
  - $F(0)$ is a real number
- If $M = 2n$, conjugate symmetric with respect to $\frac{M}{2}$: $F(\frac{M}{2} + u) = F^*(\frac{M}{2} - u)$
  - $F(\frac{M}{2})$ is a real number
- M real numbers in spatial domain are represented also by M real numbers in frequency domain

Convolution

Convolution of two M-point sequences $f(x)$ and $h(x)$:

$$\begin{aligned} g(x) &= f(x) * h(x) \\ &= \frac{1}{M} \sum_{m=0}^{M-1} f(m)h(x-m) \\ &= \frac{1}{M} \sum_{m=0}^{M-1} f(x-m)h(m) \end{aligned}$$

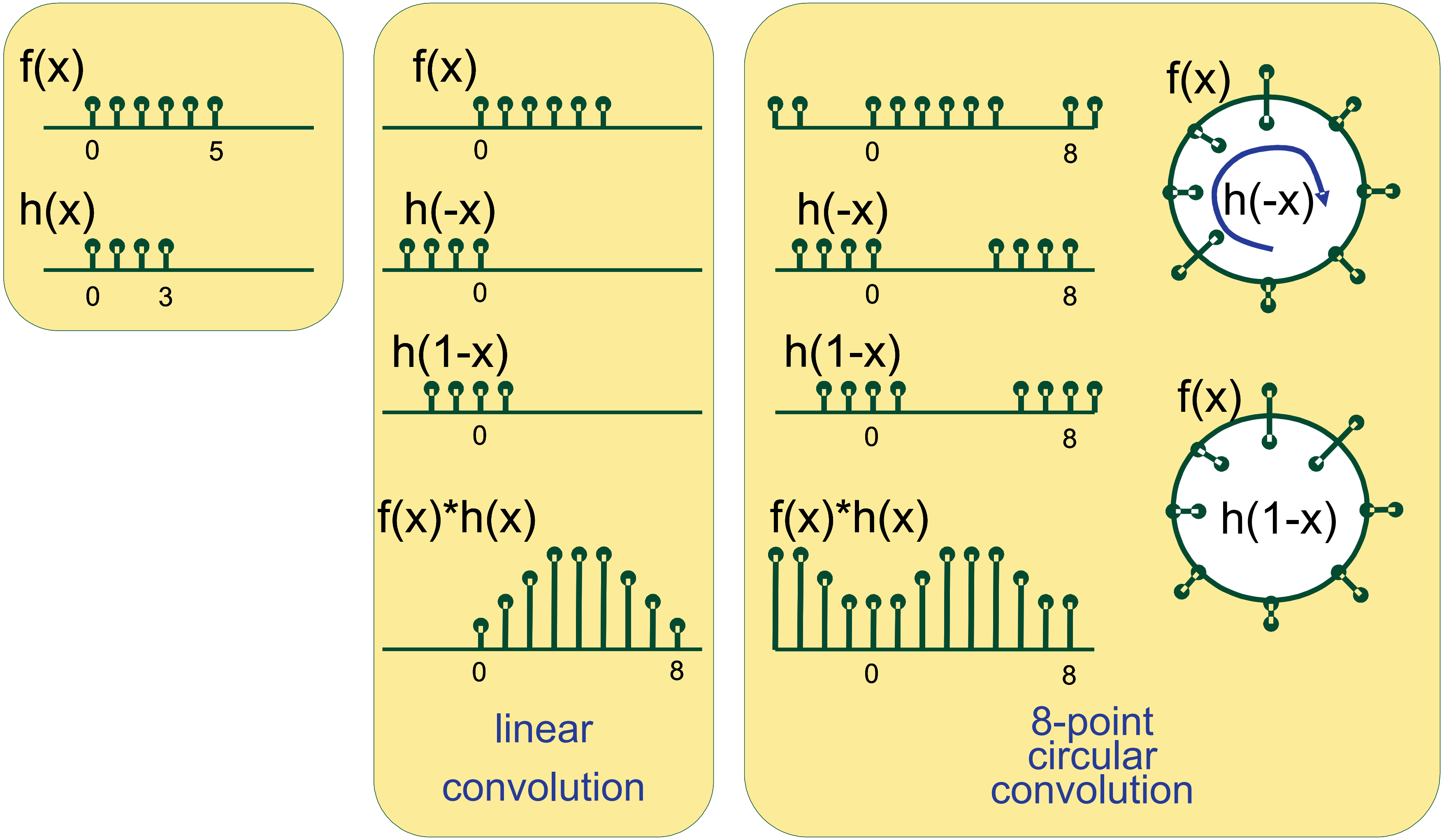
- Linear convolution

$$f(x) = g(x) = 0 \qquad \text{if } x < 0 \text{ or } x \geq M$$

- Circular convolution

$$\begin{aligned} f(x+kM) &= f(x) \\ g(x+kM) &= g(x) \end{aligned}$$

Linear convolution vs. circular convolution



Zero padding $f(x)$ and $h(x)$ to make 9-point sequences, then do 9-point circular convolution to obtain the linear convolution

Properties of DFT

P3) Using circular convolution to obtain linear convolution

Conditions

- $f(x)$: M-point sequence, $f(x) = 0$ if $x < 0$ or $x > M - 1$
- $h(x)$: N-point sequence, $h(x) = 0$ if $x < 0$ or $x > N - 1$
- Then, linear convolution of $f(x)$ and $h(x)$ will generate $(M + N - 1)$-point sequence, $f(x) * h(x) = 0$ if $x < 0$ or $x > M + N - 2$

Procedures

1. Zero padding $f(x)$ and $h(x)$ to yield $(M + N - 1)$-point sequences $f_p(x)$ and $h_p(x)$.
2. Obtain $(M + N - 1)$-point circular convolution of $f_p(x)$ and $h_p(x)$.
3. Result of Step 2 is equivalent to the linear convolution of $f(x)$ and $h(x)$.

Properties of DFT

P4) Circular convolution theorem

$$
\begin{aligned}
g(x) &= f(x) * h(x) = \frac{1}{M} \sum_{m=0}^{M-1} f(m) h(x-m) \\
\Longrightarrow G(u) &= F(u) H(u)
\end{aligned}
$$

Properties of DFT

P5) Fast Fourier transform (FFT) is available

- Complexity $= O(M \log M)$
- Circular convolution can be performed fast in the frequency domain using two FFTs and one IFFT.
- Hence, linear convolution also can be performed in the frequency domain.

Properties of DFT

P6) Computing inverse transform using forward transform

1. Take the complex conjugate of the input vector
2. Put the result of Step 1 into the forward transform as input vector
3. Take the complex conjugate of the output vector of Step 2 and multiply it by M

2D DFT

- Input image of size $M \times N$

$$f(x,y) : 0 \leq x \leq M-1,\ 0 \leq y \leq N-1$$

- Forward transform

$$\begin{aligned} F(u,v) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)} \qquad (W_M \triangleq e^{-j2\pi/M}) \\ &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)W_M^{ux}W_N^{vy} \end{aligned}$$

- Inverse transform

$$\begin{aligned} f(x,y) &= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)} \\ &= \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)W_M^{-ux}W_N^{-vy} \end{aligned}$$

Properties of 2D DFT

- 2D DFT and IDFT are separable

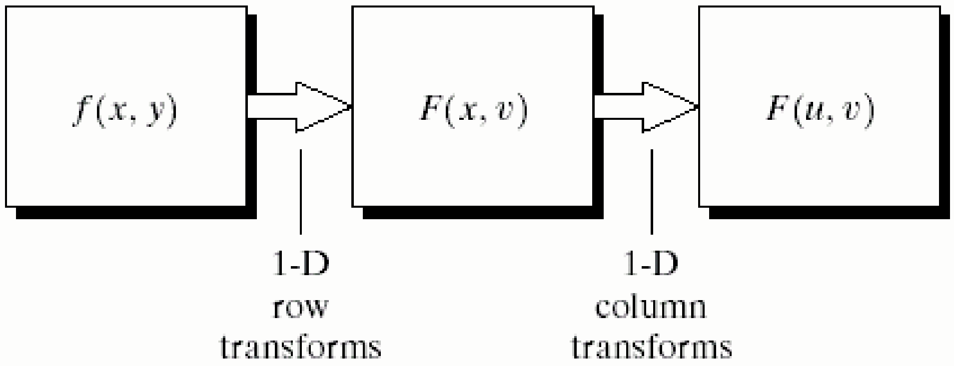


FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

Properties of 2D DFT

- Translation

$$
\begin{aligned}
f(x,y)W_M^{-u_0x}W_N^{-v_0y} &\Leftrightarrow F(u-u_0, v-v_0) \\
f(x-x_0, y-y_0) &\Leftrightarrow F(u,v)W_M^{ux_0}W_N^{vy_0}
\end{aligned}
$$

- Especially, $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$.

Properties of 2D DFT

All properties of 1D DFT also can be easily generalized to 2D case

- The extensions are periodic

$$
\begin{aligned}
f(x + kM, y + lN) &= f(x, y) \\
F(u + kM, v + lN) &= F(u, v)
\end{aligned}
$$

- The DFT of a real sequence is conjugate symmetric

$$
\begin{aligned}
F(u, v) &= F^*(-u, -v) \\
F(M/2 + u, N/2 + v) &= F^*(M/2 - u, N/2 - v)
\end{aligned}
$$

Properties of 2D DFT

All properties of 1D DFT also can be easily generalized to 2D case

- 2D circular convolution (based on the periodicity)

$$g(x,y) = f(x,y) * h(x,y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) h(x-m, y-n)$$

- $G(u,v) = F(u,v) \cdot H(u,v)$
- Obtaining linear convolution using circular convolution.
  - $f(x,y)$ of size $M \times N$
  - $h(x,y)$ of size $O \times P$

1. Zero padding f and h to make $(M+O-1) \times (N+P-1)$ images f_p and h_p
2. Linear convolution of f and h is given by

$$\mathrm{IFFT_{2D}}\left[\mathrm{FFT_{2D}}[f_p] \times \mathrm{FFT_{2D}}[g_p]\right]$$