0/1 Knapsack Problem

Maximize $\sum_{i=1}^{N} p_i x_i$ when $\sum_{i=1}^{N} w_i x_i \le C$

$x_i = 0$ or 1 for $1 \le i \le N$.

Example

i	1	2	3	4	5
w_i	4	3	5	6	2
p_i	9	8	10	9	3

$N = 5$

$C = 8$

Make it easier by considering the first k items only.

How can we relate k-item problem with $(k+1)$-item problem?

If we choose $(k+1)$th item, the reduced problem has capacity $- w_{k+1}$

do not choose $(k+1)$th item ,,

capacity.

(k, α) Items $(1, 2, \cdots, k)$ are available with sub-capacity α

$f(k, \alpha)$ the maximum profit for (k, α) - problem state.

k \ α	0	1	2	3	4	5	6	7	8	
1	0	0	0	0	9	9	9	9	9	N
2	0	0	0	8	9	9	9	17	17	Y
3	0	0	0	8	9	9	9	17	18	Y
4	0	0	0	8	9	9	9	17	18	No
5	0	0	3	8	9	11	12	17	18	No

$$f(k,\alpha) = \begin{cases} 0 & \alpha < w_1 \\ p_1 & \alpha \geq w_1 \end{cases}$$

$$f(2,\alpha) = \begin{cases} f(1,\alpha) & \alpha < w_2 \\ \max\left[\begin{matrix} f(1,\alpha - w_2) + p_2 \\ f(1,\alpha) \end{matrix}\right] \end{cases}$$

$$= \max\left(\begin{matrix} f(1,\alpha-3)+8 \\ f(1,\alpha) \end{matrix}\right)$$

k=3

$$f(3,\alpha) = \begin{cases} f(2,\alpha) & \alpha < 5 \\ \max\{f(2,\alpha-5)+10,\ f(2,\alpha)\} & \alpha \geq 5 \end{cases}$$

k=4

$$f(4,\alpha) = \begin{cases} f(3,\alpha) & \alpha < 6 \\ \max(f(3,\alpha-6)+9,\ f(3,\alpha)) & \alpha \geq 6 \end{cases}$$

k=5

$$f(5,\alpha) = \begin{cases} f(4,\alpha) & \alpha < 2 \\ \max(f(4,\alpha-2)+3,\ f(4,\alpha)) & \alpha \geq 2 \end{cases}$$

Traceback

items 2 and 3 are selected.

Matrix Multiplication

A B

$m \times n$ $n \times q$

$$m \begin{array}{c} n \\ \left[\quad \right] \end{array} n \begin{array}{c} q \\ \left[\quad \right] \end{array}$$

$n \times (mq)$

$= mnq$ multiplications are required.

Ex 1) A B C

100×1 1×100 100×1

① (A B) C

$T = AB \Rightarrow$ 10,000

TC 1,000,000

Total 1,010,000

② A (B C)

$S = BC \Rightarrow$ 100

AS 100

Total. 200

Ex 2)

A B C D E

① ② ③ ④

There are about 4 ! methods

1 2 3 4 $\{[(AB)C]D\}E$

1 2 4 3 $[(AB)C] \times (DE)$ = 4 1 2 3

1 3 2 4

1 3 4 2

⋮

Ex 3) M_1 M_2 M_3 M_4 M_5 M_6

dim. $r_1 \times r_2$ $r_2 \times r_3$ $r_3 \times r_4$ $r_4 \times r_5$ $r_5 \times r_6$ $r_6 \times r_7$

How can we find the best method with the least # of 'x's?

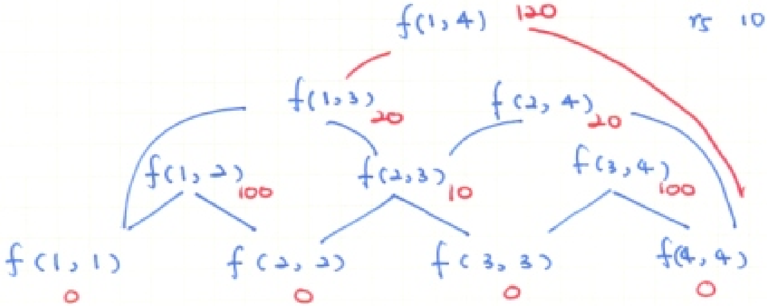
$f(i, j)$: the least # of 'x''s required to compute $M_i M_{i+1} \cdots M_j$

$$f(1,6) = \min \begin{cases} f(1,1) + f(2,6) + r_1 r_2 r_7 \\ f(1,2) + f(3,6) + r_1 r_3 r_7 \\ \vdots \\ f(1,5) + f(6,6) + r_1 r_6 r_7 \end{cases}$$

Ex 4) M_1 M_2 M_3 M_4

10×1 1×10 10×1 1×10

r_1	10
r_2	1
r_3	10
r_4	1
r_5	10



$$f(1,3) = \min \begin{cases} \boxed{f(1,1) + f(2,3) + r_1 r_2 r_4} \\ f(1,2) + f(3,3) + r_1 r_3 r_4 \end{cases}$$

$[M_1 \times (M_2 \times M_3)] \times M_4$

$\Uparrow$

$$f(2,4) = \min \begin{cases} f(2,2) + f(3,4) + r_2 r_3 r_5 \\ f(2,3) + f(4,4) + r_2 r_4 r_5 \end{cases}$$

$$= \min \begin{cases} f(1,1) + f(2,4) + r_1 \overset{100}{r_2} r_5 \\ f(1,2) + f(3,4) + r_1 \overset{1000}{r_3} r_5 \\ \boxed{f(1,3) + f(4,4) + r_1 \overset{100}{r_4} r_5} \end{cases}$$

M_1 M_2 M_3 M_4

x

x

x

1. (20 points) In a country, they have five kinds of coins: 1-cent, 5-cent, 10-cent, 21-cent, and 25-cent coins. Kim is working in a convenience store, and needs to make K cents of change using the minimum number of coins. For example, to make 23 cents of change, the minimum number is three (i.e. one 21-cent coin and two 1-cent coins).
   (a) What is the minimum number to make 62 cents of change?
   (b) Describe a dynamic programming to return the minimum number for a general problem of making K cents of change.

$d(k)$ = min # to make k cents

$$d(k) = \min \begin{cases} d(k-1) + 1 \\ d(k-5) + 1 \\ d(k-10) + 1 \\ d(k-21) + 1 \\ d(k-25) + 1 \end{cases}$$

k	1	2	3	4	5	6	7	8	...

?

(a) 찍어서 4개? (21, 21, 10, 10)

initial condition

(b) $d(k) = \min \begin{Bmatrix} d(k-1) \\ d(k-5) \\ \vdots \\ d(k-25 \end{Bmatrix} + 1$

$d(k) = \infty \quad k < 0$

$d(0) = 0$

k는 1 부터 K까지 위의 식을 실행한 후

trace back