Digital Signal Processing

Chap 3. The z-Transform

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Definitions

z-Transform

- z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Ex) $x[n] = \delta[n+1] + 2\delta[n] - 3\delta[n-2]$

$$\Rightarrow \quad X(z) = z + 2 - 3z^{-2}$$

- z-Transform is simply an alternative representation of a signal
  - The coefficient of z^{-n} is the signal value $x[n]$

z-Transform

- Ex) z-transform pair

$$\frac{1}{2^n}u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}$$

ROC
(region of convergence)

z-Transform is an extension of DTFT

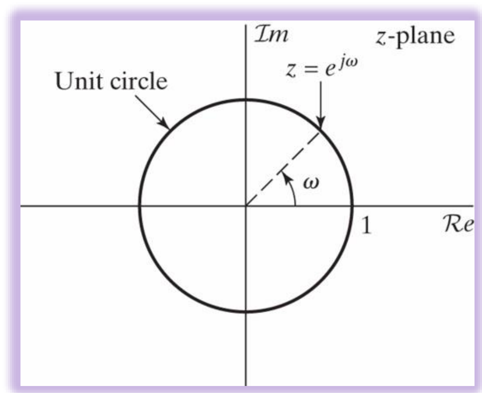
- z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- z-Transform vs. DTFT
  - DTFT of $x[n] = X(z)|_{z=e^{j\omega}}$



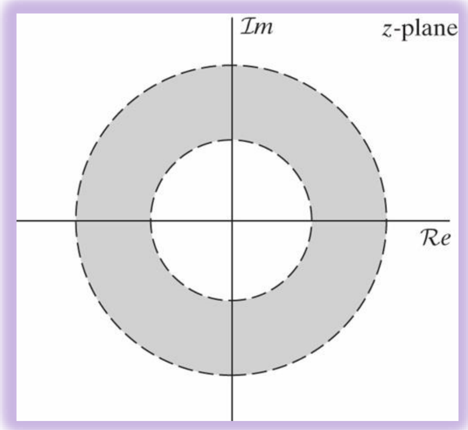
z-Transform and DTFT

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- If $z = re^{j\omega}$

$$X(z) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n}$$

which is DTFT of $x[n]r^{-n}$.



- Convergence of DTFT
  - Is $x[n]$ absolutely summable?
- Convergence of z-Transform
  - Is $x[n]r^{-n}$ absolutely summable?
  - Therefore, the region of convergence will be a ring shape.

Why do we need the extension?

- Consider the DTFT pair

$$a^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1 - ae^{-jw}} \qquad |a| < 1$$

What happens if |a| ≥ 1?

- z-Transform pair

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

ROC
(region of convergence)

- z-Transform can be applied to a broader class of signals than DTFT
  - It is useful in studying a broader class of systems
  - It is used to analyze the causality and stability of a system

ROC should be specified

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| > |a|$$

$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \qquad |z| < |a|$$

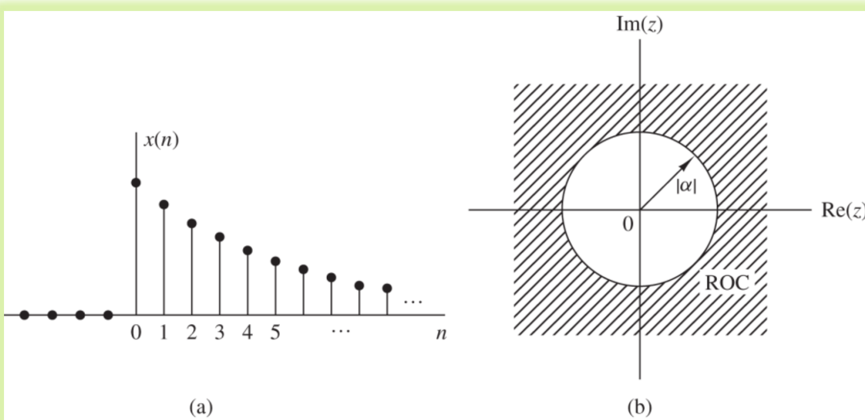


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

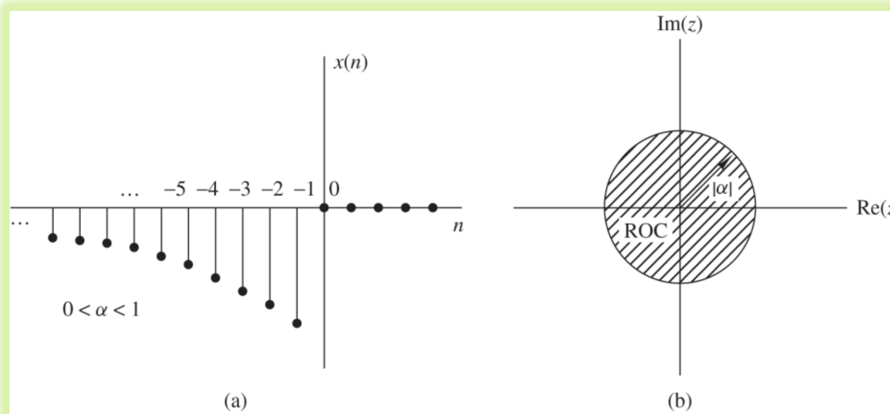
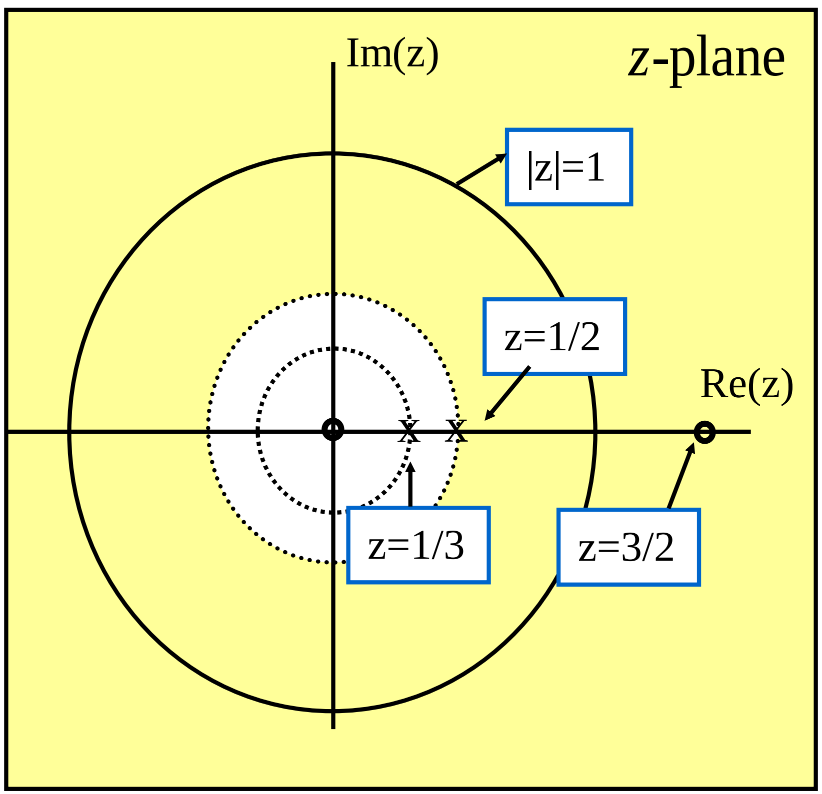


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

ROC should be specified

Ex) $$x[n] = 7(\frac{1}{3})^n u[n] - 6(\frac{1}{2})^n u[n]$$

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| > \frac{1}{2}$$

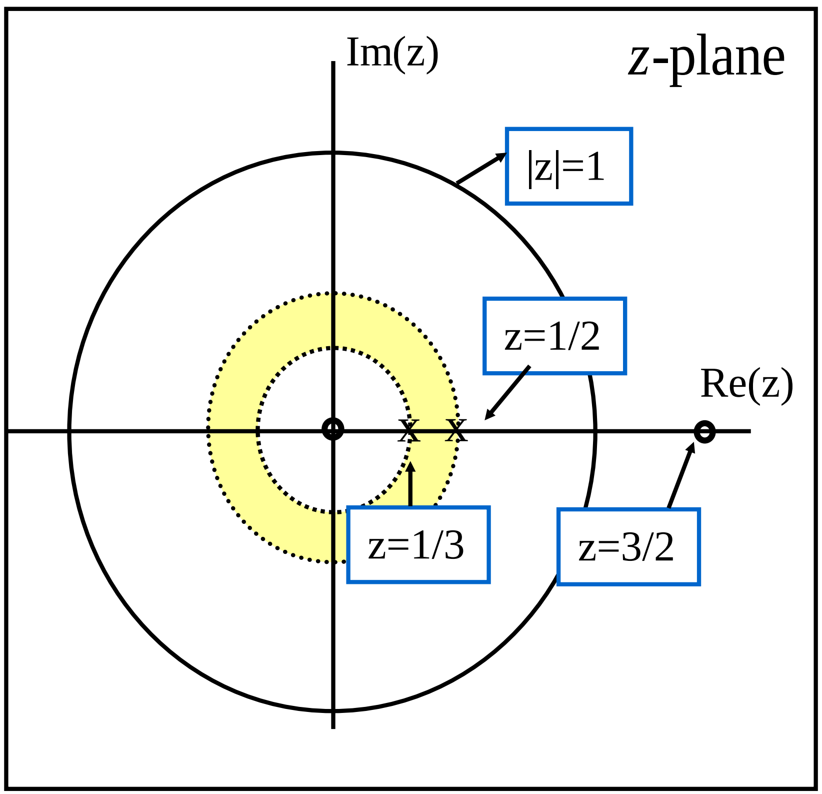


There are other sequences, which generate the same $X(z)$ but with different ROC's

ROC should be specified

Ex) $x[n] = ?$

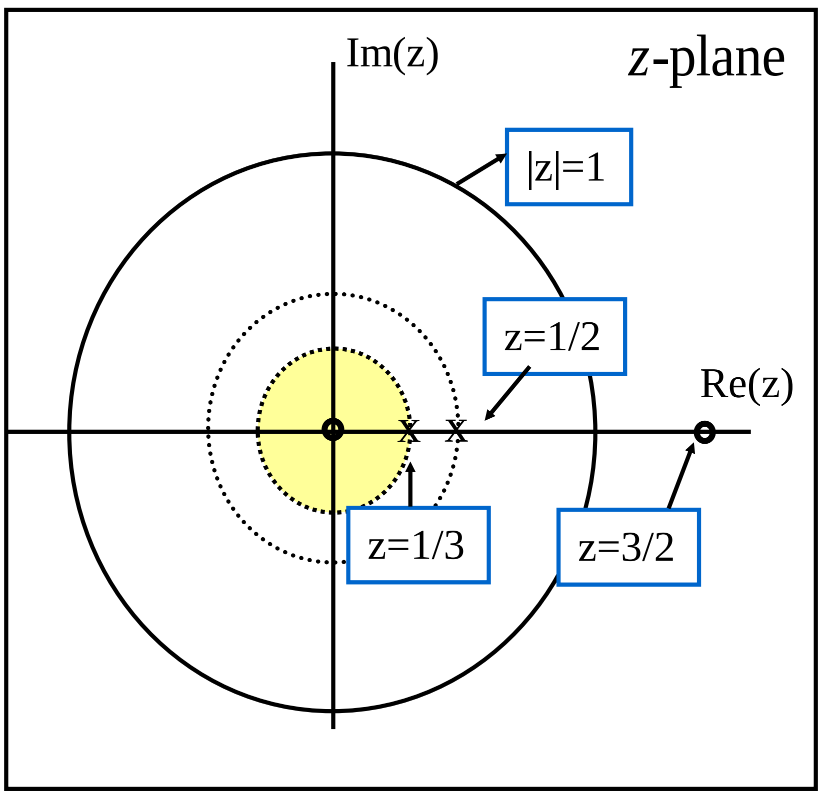
$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$



ROC should be specified

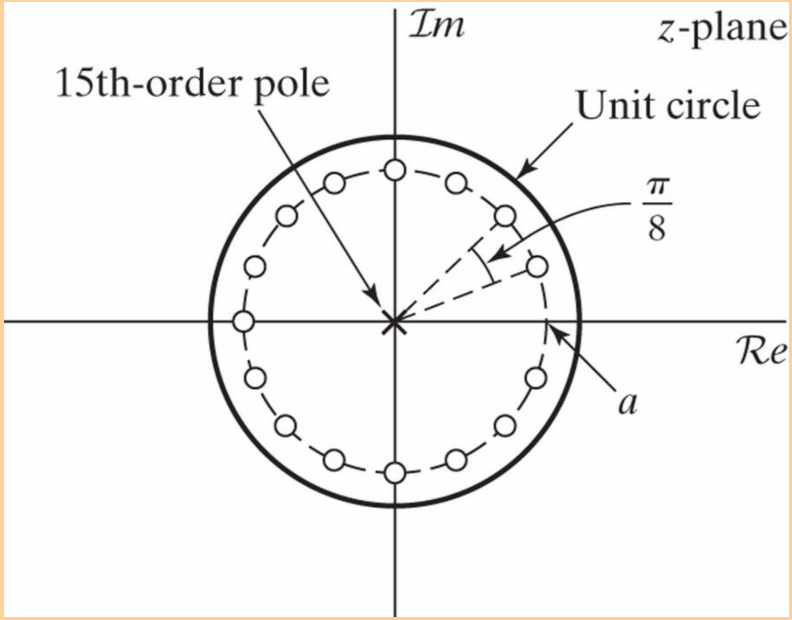
Ex) $x[n] = ?$

$$X(z) = \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}, \quad |z| < \frac{1}{3}$$



Another Example

- $x[n] = \begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise.} \end{cases}$



Pole–zero plot when $N = 16$ and a is real such that $0 < a < 1$. The ROC in this example consists of all values of z except $z = 0$.

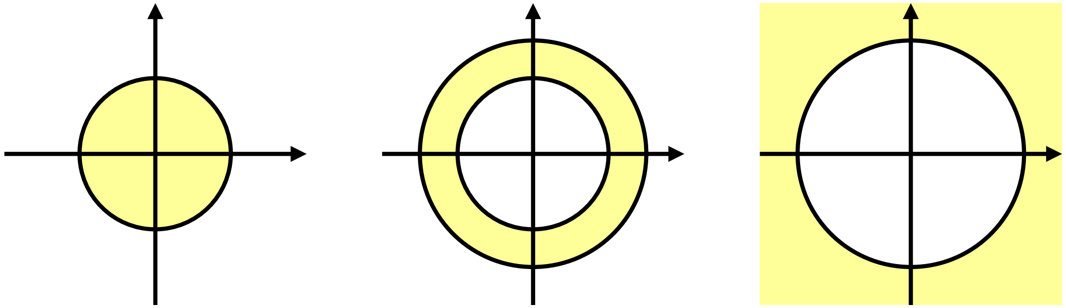
Common z-Transform Pairs

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $r^n\cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $r^n\sin(\omega_0 n)u[n]$	$\frac{r\sin(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0 \le n \le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Properties on ROC

- ROC of $X(z)$ consists of a single ring in the z-plane centered at the origin
  - Proof: Skipped. Refer to any textbook on complex analysis.

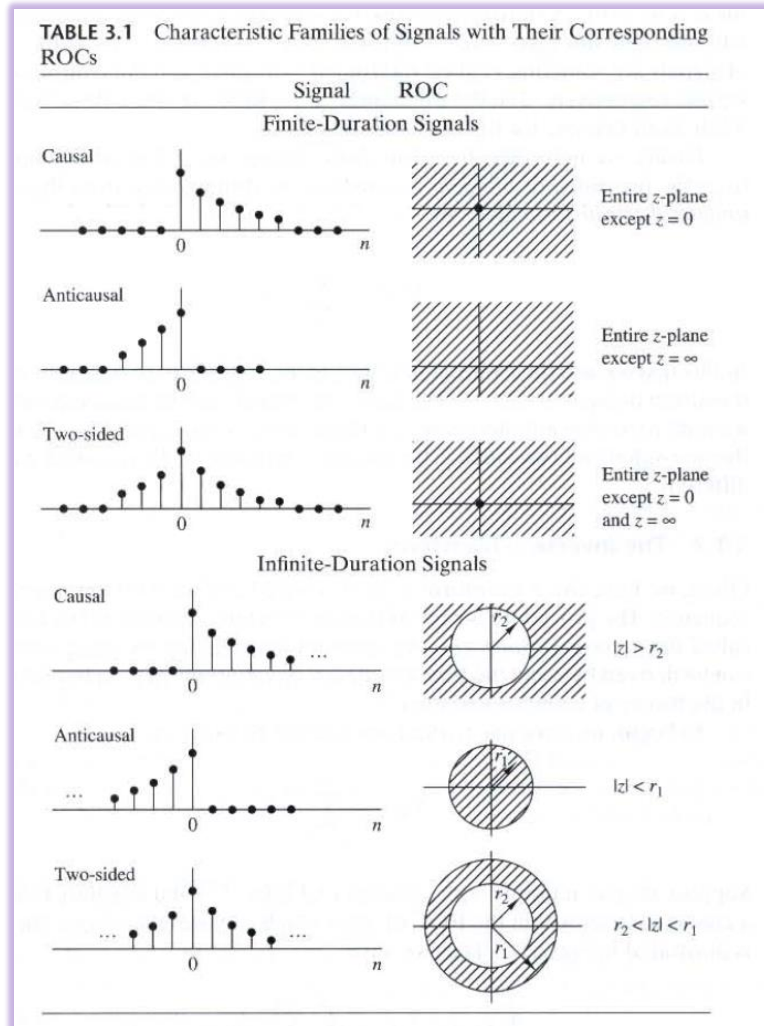


- ROC does not contain any poles.
- $x[n]$ has the Fourier transform, if ROC includes the unit circle

Properties on ROC

- Suppose that $X(z)$ is rational.
  - $x[n]$ is a finite duration sequence
    $\Rightarrow$ ROC is the entire z-plane
    except possibly $z = 0$ or $z = \infty$.
  - $x[n]$ is right-sided
    $\Rightarrow$ ROC is the region in the z-plane
    outside the outermost pole.
  - $x[n]$ is left-sided
    $\Rightarrow$ ROC is the region inside the
    innermost nonzero pole.
  - $x[n]$ is two-sided
    $\Rightarrow$ ROC is a ring, bounded on the
    interior and the exterior by poles.

- ROC is a connected region

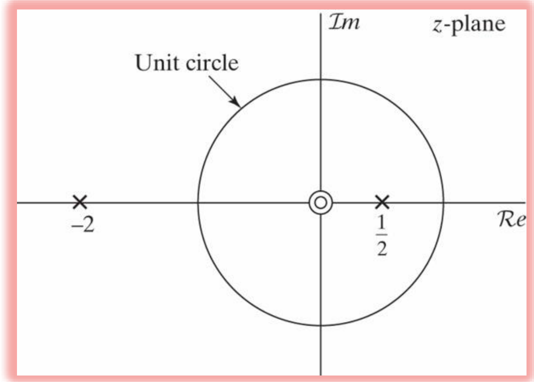


Some signals don't have z-transforms

- $x[n] = \frac{1}{2^n} u[n] - \left(-\frac{1}{3}\right)^n u[-n-1].$

Analysis of LTI Systems in z-Domain

- Causality
  - ROC of the system function is the exterior of a circle



- Stability
  - ROC contains the unit circle
- A causal system is stable if all poles are inside the unit circle

Inverse z-Transforms

Inverse z-Transform

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- C is any closed contour within the ROC of the z-transform
- Its proper evaluation requires some knowledge on complex integral
  - For example, you may refer to R. V. Churchill and J. W. Brown, *Complex Variables and Applications*, McGraw-Hill
- We do not use this formula. Instead, we decompose $X(z)$ into a number of terms, each of which can be inverse transformed using tables or partial fractions

Inverse z-Transform by Partial-Fraction Expansion

Ex 1) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Inverse z-Transform by Partial-Fraction Expansion

Ex 2) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1+z^{-1}}{1-z^{-1}+0.5z^{-2}}$$

Inverse z-Transform by Partial-Fraction Expansion

Ex 3) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

Inverse z-Transform by Partial-Fraction Expansion

Ex 4) Determine the causal signal $x[n]$, whose z-transform is

$$X(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$$

Inverse z-Transform by Power Series Expansion

Ex 5) Determine the signal $x[n]$, whose z-transform is

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|.$$

Properties

Properties of z-Transform

- Linearity

If $x_1[n] \Leftrightarrow X_1(z)$, ROC=$R_{x_1}$

$x_2[n] \Leftrightarrow X_2(z)$, ROC=$R_{x_2}$

Then

$$a\,x_1[n] + b\,x_2[n] \Leftrightarrow a\,X_1(z) + b\,X_2(z),$$

ROC contains $R_{x_1} \cap R_{x_2}$

Ex) Determine the z-transforms of

$\cos(\omega_0 n)u[n]$ and $\sin(\omega_0 n)u[n]$

Ex) Determine the z-transform of

$x[n] = a^n(u[n] - u[n-N])$.

Properties of z-Transform

- Time shifting

$$x[n-k] \iff z^{-k}X(z),$$

  - ROC=R_x

    (except for the possible addition or deletion of $z=0$ or $z=\infty$)

Ex) Determine the inverse z-transform of

$$X(z)=\frac{1}{z-\frac{1}{4}},\ \ |z|>\frac{1}{4}$$

Properties of z-Transform

- Scaling in the z-domain

$$a^n x[n] \Leftrightarrow X(z/a), \quad \text{ROC}=|a|R_x.$$

Ex) Determine the z-transform of

$$x[n] = r^n \cos(\omega_0 n)u[n]$$

Properties of z-Transform

- Differentiation in the z-domain

$$nx[n] \Leftrightarrow -z\frac{dX(z)}{dz}, \quad \text{ROC}=R_x.$$

Ex) Determine the z-transform of $na^n u[n]$

Ex) Determine the signal $x[n]$ corresponding to

$$X(z) = \log(1 + az^{-1}),\ |z| > |a|$$

Properties of z-Transform

- Time reversal

$$x[-n] \Leftrightarrow X(\frac{1}{z}), \quad \text{ROC}=\frac{1}{R_x}.$$

Ex) Determine the z-transform of $a^{-n}u[-n]$

Properties of z-Transform

- Convolution becomes multiplication

$$x_1[n] * x_2[n] \Leftrightarrow X_1(z)X_2(z),$$

ROC contains $R_{x_1} \cap R_{x_2}$

Ex) Compute the convolution of

$$x_1[n] = \delta[n] + 2\delta[n-1] + \delta[n-2] \quad \text{and}$$
$$x_2[n] = \delta[n] - \delta[n-1].$$

Properties of z-Transform

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	Contains $R_{x_1} \cap R_{x_2}$
2	3.4.2	$x[n - n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\|z_0\|R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{R}e\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Contains R_x
7		$\mathcal{I}m\{x[n]\}$	$\frac{1}{2j}[X(z) - X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1} \cap R_{x_2}$

z-Transform and LTI Systems

System Function

- A system function is the z-transform of an impulse response

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = \frac{Y(z)}{X(z)}$$

- If the system is given by CCDE

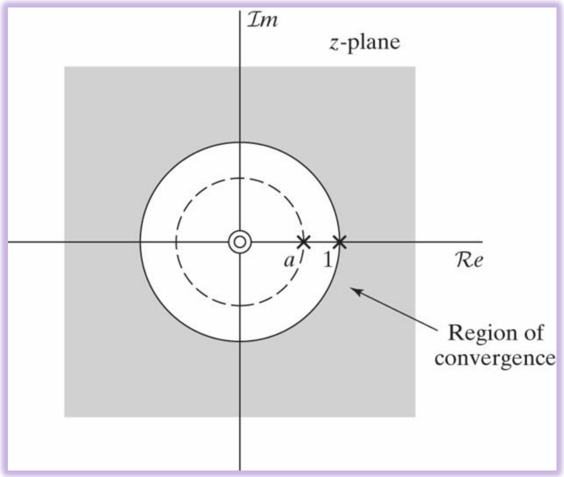
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

then

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Example

- $h[n] = a^n u[n]$ and $x[n] = Au[n]$.



Example

- $y[n] = ay[n-1] + x[n]$.

- We will see more applications of the z-transform in Chapter 5.

Analysis of LTI Systems in z-Domain

Ex) An LTI system is characterized by the system function

$$H(z) = \frac{1}{1 - 0.5z^{-1}} + \frac{2}{1 - 3z^{-1}}.$$

Specify the ROC of $H(z)$ and determine $h[n]$ for the following conditions

a. The system is stable
b. The system is causal

Unilateral z-Transform

Unilateral z-Transform and Its Properties

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- $y[n] = x[n-1]$
  $$\Leftrightarrow \ \mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z)$$
- $y[n] = x[n-2]$
  $$\Leftrightarrow \ \mathcal{Y}(z) = x[-2] + x[-1]z^{-1} + z^{-2}\mathcal{X}(z)$$
- and so forth

Example

$$y[n] - ay[n-1] = x[n] \text{ and } x[n] = u[n].$$

- Note that CCDE describes an LTI system only if we assume initial rest conditions.
- But, in this example, we assume $y[-1] \neq 0$.

$$y[n] = \begin{cases} y[-1], & n = -1, \\ y[-1]a^{n+1} + \dfrac{1}{1-a}(1 - a^{n+1}), & n \geq 0. \end{cases}$$

zero input response zero initial condition response