Digital Signal Processing

Quantization

Chang-Su Kim

Quantization

- Digitization =
  - sampling (coordinate) + quantization (value)

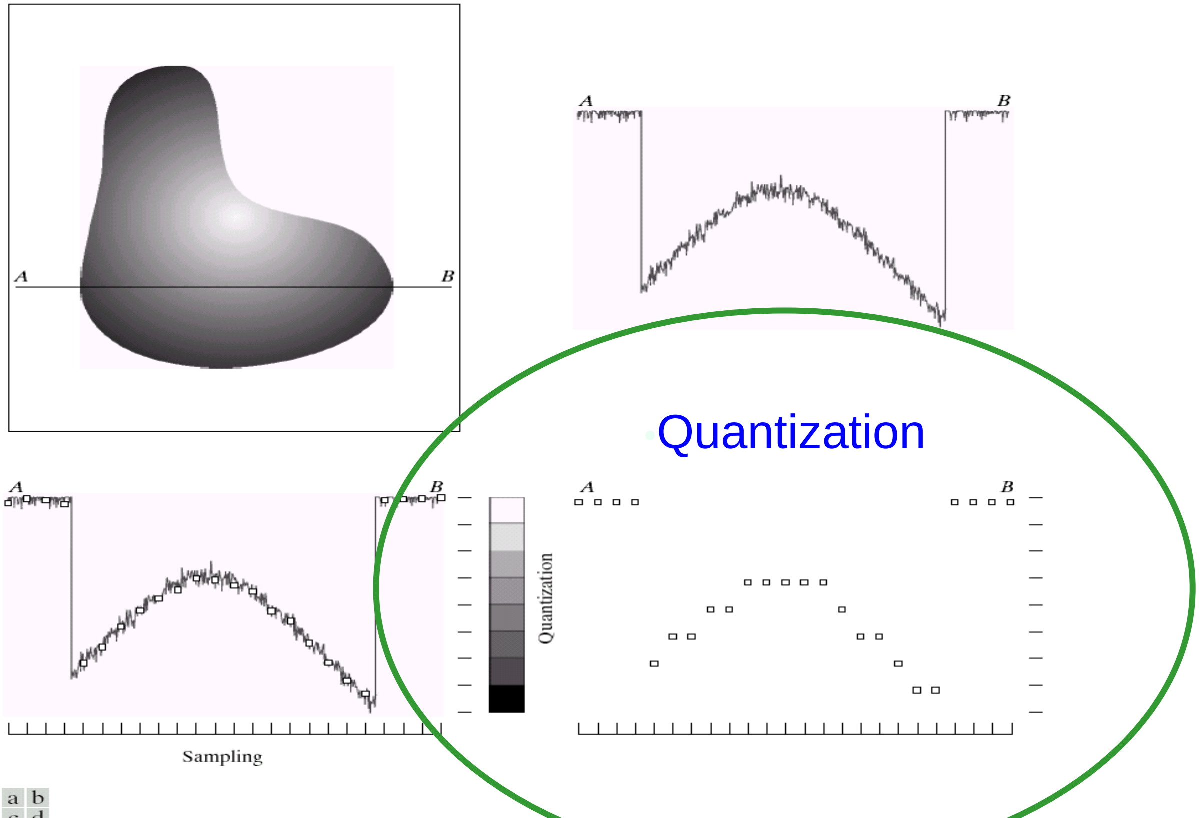
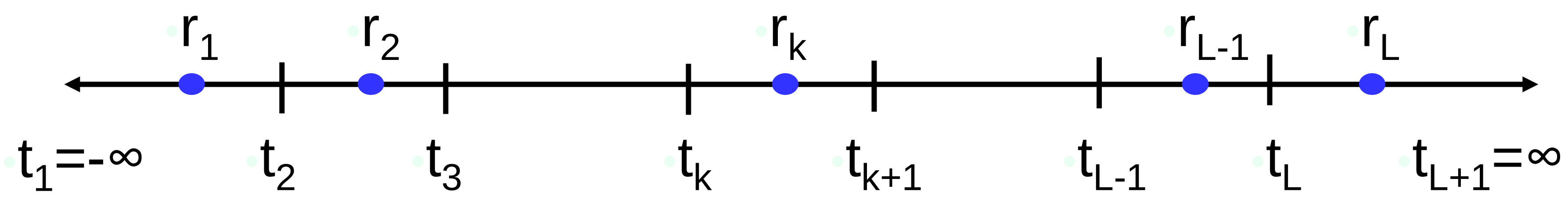


FIGURE 2.16 Generating a digital image. (a) Continuous image. (b) A scan line from A to B in the continuous image, used to illustrate the concepts of sampling and quantization. (c) Sampling and quantization. (d) Digital scan line.

Quantizer

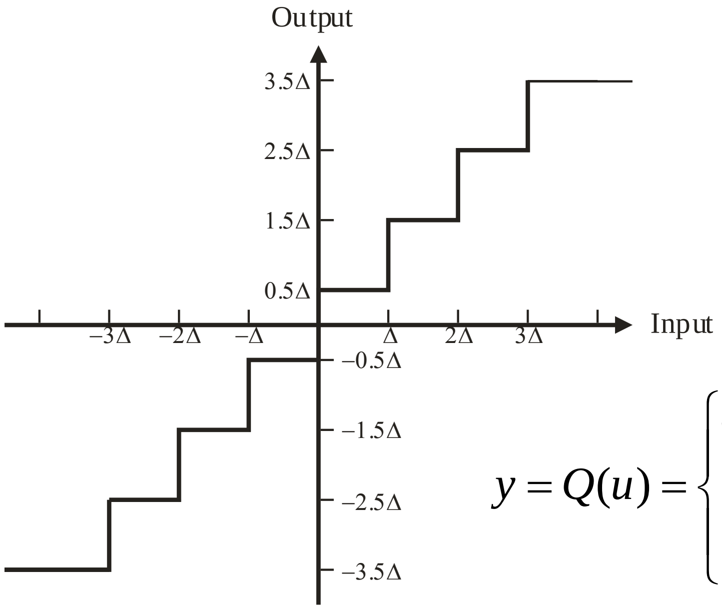
- A quantizer Q maps a continuous variable u into a discrete variable Q(u) in $\{r_1, r_2, r_3, \ldots, r_L\}$



- Partition the real line into L cells and map input values within a cell into a constant r_k
  - $Q(u) = r_k \quad \text{if} \;\; t_k \leq u < t_{k+1}$
  - r_k : reconstruction level
  - t_k : transition or decision level
  - $\Delta_k = t_{k+1} - t_k$: step size

Quantizer Example

- Input-output graph of an 8-level quantizer



$$y = Q(u) = \begin{cases} 3.5\Delta & \text{if } u > 3\Delta, \\ 0.5(2n-1)\Delta & \text{if } (n-1)\Delta < u \le n\Delta \ (n = -2, -1, \ldots, 3), \\ -3.5\Delta & \text{if } u \le -3\Delta. \end{cases}$$

- Uniform quantizer
  - Except the outer two cells
    - $t_{k+1} - t_k = \Delta$ and $r_k = (t_k + t_{k+1})/2$

Lloyd-Max Quantizer

- Quantization error: $u - Q(u)$
- Probability distribution of input: $p(u)$
- Mean square error (MSE)

$$\mathcal{E} = E[(u - Q(u))^2] = \int_{t_1}^{t_{L+1}} (u - Q(u))^2 p(u) du$$

- Lloyd-Max quantizer minimizes $\mathcal{E}$,
  i.e. it is the minimum mean square error (MMSE) quantizer

Lloyd-Max Quantizer – Centroid Condition

- MSE

$$\mathcal{E} = \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} (u - Q(u))^2 p(u) du = \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} (u - r_i)^2 p(u) du$$

- For fixed transition levels t_k's, find the optimum reconstruction levels r_k's
- Minimize each $\int_{t_k}^{t_{k+1}} (u - r_k)^2 p(u) du$

$$\begin{aligned}
\mathcal{E}_k &= \int_{t_k}^{t_{k+1}} (u - r_k)^2 p(u) du \\
&= \int_{t_k}^{t_{k+1}} u^2 p(u) du - 2r_k \int_{t_k}^{t_{k+1}} up(u) du + r_k^2 \int_{t_k}^{t_{k+1}} p(u) du \\
\therefore \quad \frac{\partial \mathcal{E}_k}{\partial r_k} &= -2 \int_{t_k}^{t_{k+1}} up(u) du + 2r_k \int_{t_k}^{t_{k+1}} p(u) du = 0 \\
\therefore \quad r_k &= \frac{\int_{t_k}^{t_{k+1}} up(u) du}{\int_{t_k}^{t_{k+1}} p(u) du} = E[u | u \in [t_k, t_{k+1})]
\end{aligned}$$

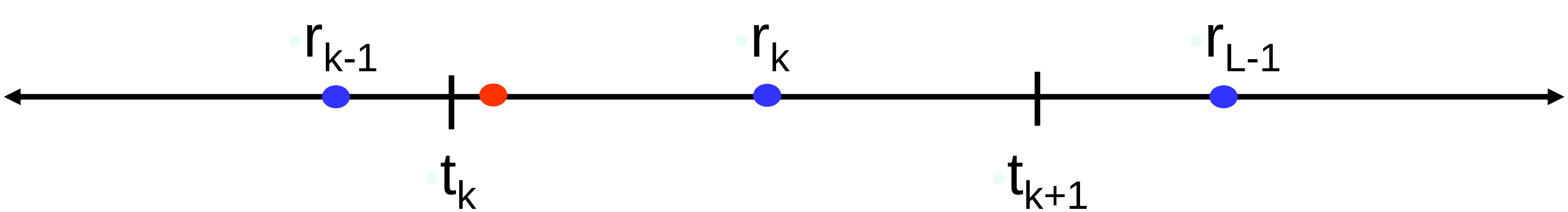
- This is called the centroid (center of mass) condition

Lloyd-Max Quanitzer – NN condition

- For fixed r_k's, find the optimum t_k's

$$
\begin{aligned}
\frac{\partial}{\partial t_k}\mathcal{E} &= \frac{\partial}{\partial t_k}\sum_{i=1}^{L}\int_{t_i}^{t_{i+1}}(u-r_i)^2 p(u)du \\
&= \frac{\partial}{\partial t_k}\left(\ldots + \int_{t_{k-1}}^{t_k}(u-r_{k-1})^2 p(u)du + \int_{t_k}^{t_{k+1}}(u-r_k)^2 p(u)du + \ldots\right) \\
&= (t_k - r_{k-1})^2 p(t_k) - (t_k - r_k)^2 p(t_k) = 0 \\
& \left(\because \frac{\partial}{\partial \alpha}\int_{\beta}^{\alpha} f(x)dx = f(\alpha), \frac{\partial}{\partial \alpha}\int_{\alpha}^{\beta} f(x)dx = -f(\alpha)\right) \\
\therefore \quad & (t_k - r_{k-1})^2 = (t_k - r_k)^2 \\
\therefore \quad & t_k = \frac{r_{k-1} + r_k}{2}
\end{aligned}
$$

- This is called the nearest neighbor condition



Design of Lloyd-Max Quantizer

- Centroid condition

$$r_k = \frac{\int_{t_k}^{t_{k+1}} u p(u) du}{\int_{t_k}^{t_{k+1}} p(u) du}$$

- Nearest neighbor condition

$$t_k = \frac{r_{k-1} + r_k}{2}$$

- These two conditions are iteratively applied to obtain the optimal quantizer

Lloyd-Max Quantizer for Uniform Distribution

$$p(u) = \begin{cases} \frac{1}{t_{L+1}-t_1}, & t_1 < u < t_{L+1} \\ 0, & \text{otherwise} \end{cases}$$

- The input has variance $\sigma_u^2 = A^2/12$, where $A = t_{L+1} - t_1$.
- From the centroid condition

$$r_k = \frac{\int_{t_k}^{t_{k+1}} up(u)du}{\int_{t_k}^{t_{k+1}} p(u)du} = \frac{t_{k+1}^2 - t_k^2}{2(t_{k+1} - t_k)} = \frac{t_{k+1} + t_k}{2} \quad (1)$$

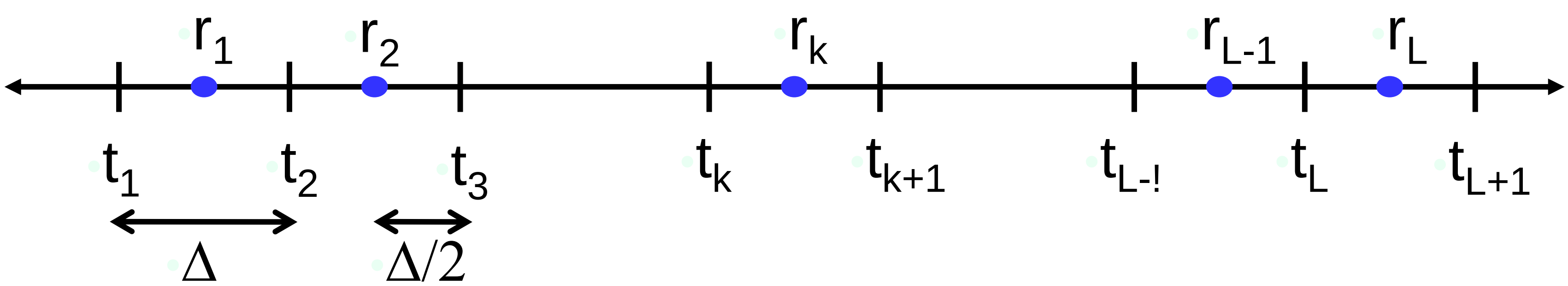
- Also, the nearest neighbor condition is

$$t_k = \frac{r_{k-1} + r_k}{2} \quad (2)$$

- By inserting (1) into (2), we have

$$\begin{aligned} t_k &= \frac{t_k + t_{k-1} + t_{k+1} + t_k}{4} \\ \Rightarrow \quad & t_k - t_{k-1} = t_{k+1} - t_k = \text{constant} \doteq \Delta \end{aligned}$$

Lloyd-Max Quantizer for Uniform Distribution



- The quantization error $\eta = u - Q(u)$ is uniformly distributed over $[-\Delta/2, \Delta/2)$.

$$\mathcal{E} = E[\eta^2] = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} x^2 dx = \frac{\Delta^2}{12}$$

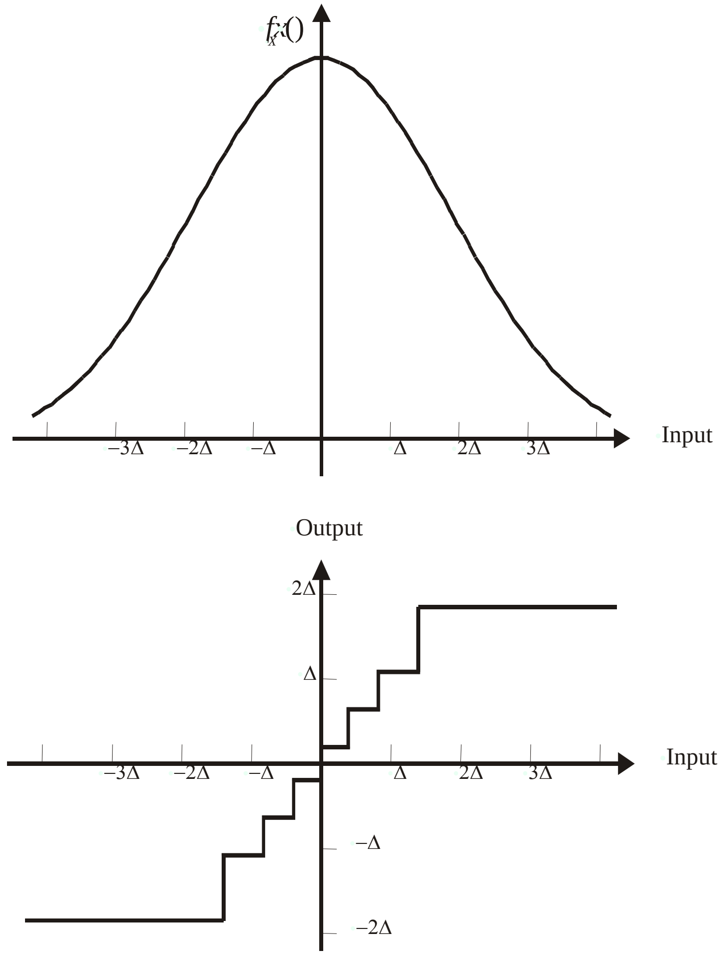
- If the quantization resolution is B bits,

$$\Delta = \frac{A}{2^B}$$

- Thus, SNR is given by

$$\begin{aligned} 10\log_{10}\frac{\sigma_u^2}{\mathcal{E}} &= 10\log_{10}\frac{A^2/12}{\Delta^2/12} \\ &= 10\log_{10} 2^{2B} = 20B\log_{10} 2 \simeq 6B \quad \text{(dB)} \end{aligned}$$

Lloyd-Max Quantizer for Other Distributions

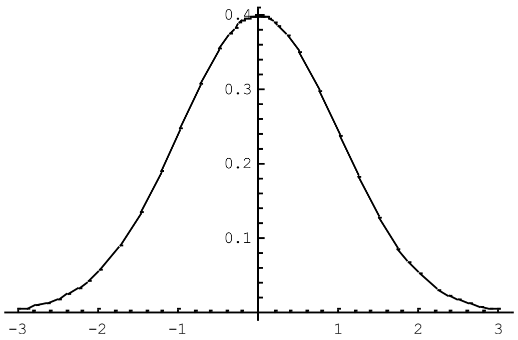


- Notice that the Lloyd-Max quantizer reduces the average distortion by approximating the input more precisely in regions of higher probability.

Lloyd-Max Quantizer for Other Distributions

- Gaussian: for pixel distribution

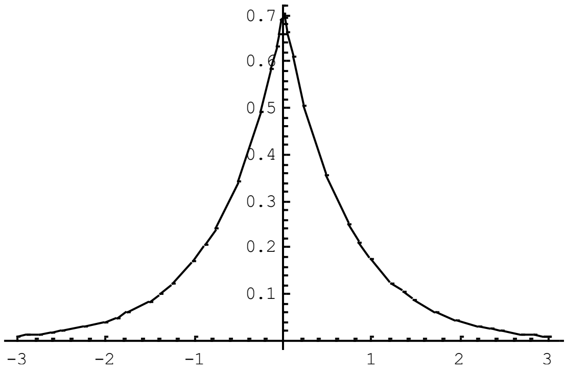
$$p(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\frac{-(u-\mu)^2}{2\sigma^2})$$



- Laplacian: for the distribution of differences between adjacent pixels

$$p(u) = \frac{\lambda}{2} \exp(-\lambda|u-\mu|)$$

where $\sigma^2 = \frac{2}{\lambda^2}$



- Look-up table of Lloyd-Max Q is available for these distributions

K - Means

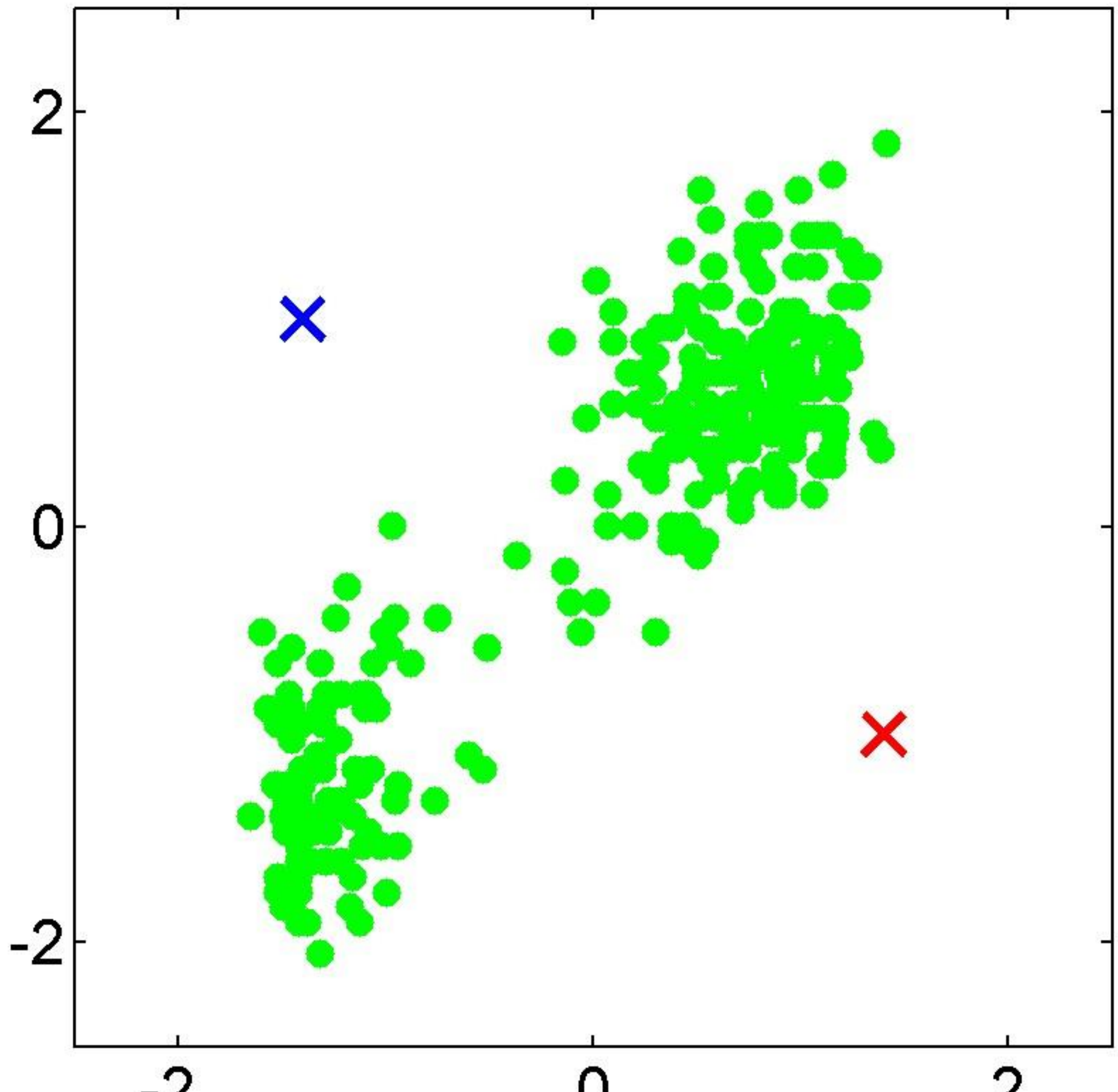
K-Means Algorithm

Choose k data points to act as cluster centers

Until the cluster centers are unchanged

- Allocate each data point to cluster whose center is nearest (NN rule)
- Replace the cluster centers with the mean of the elements in their clusters (centroid rule)

End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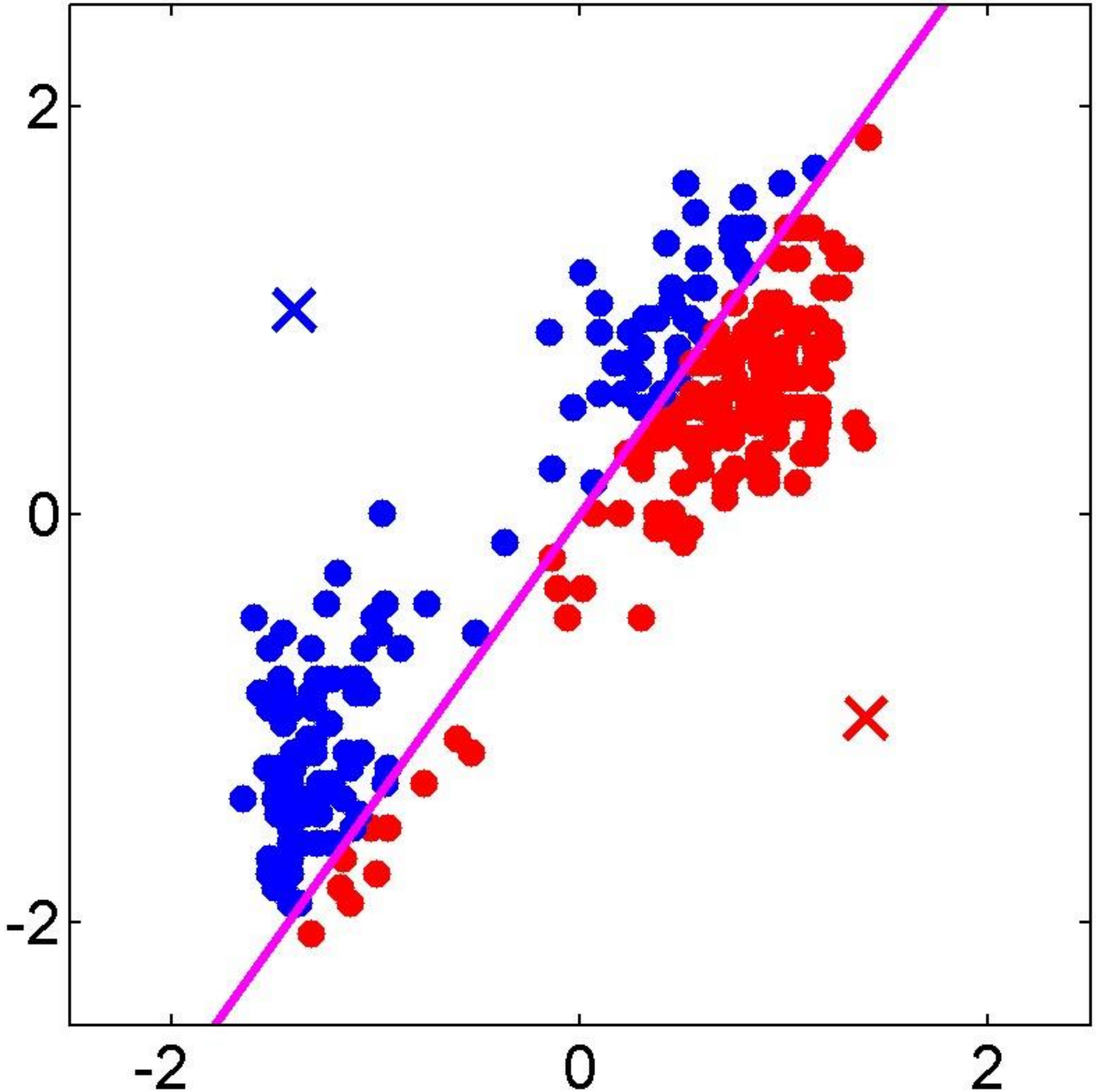

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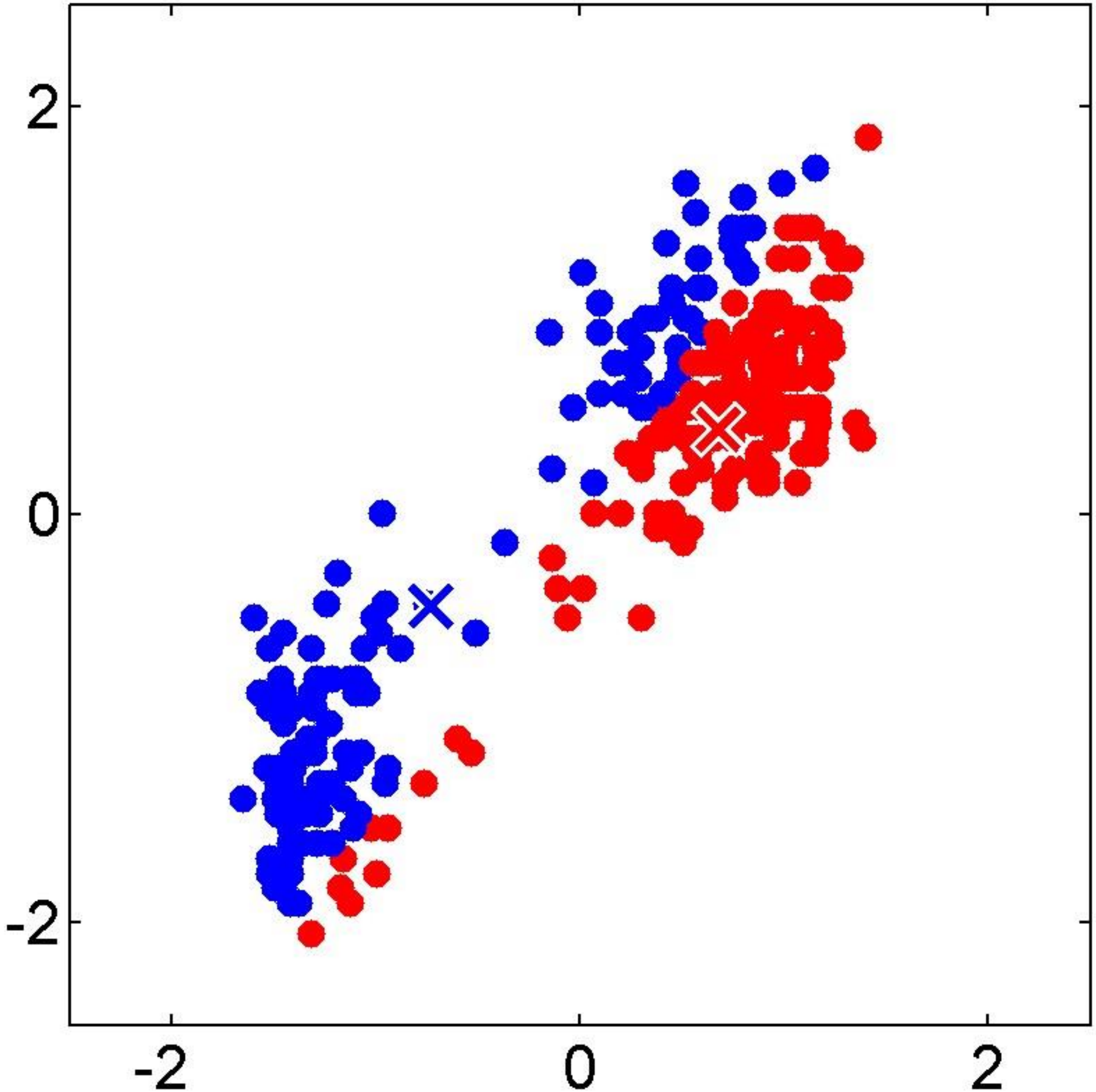
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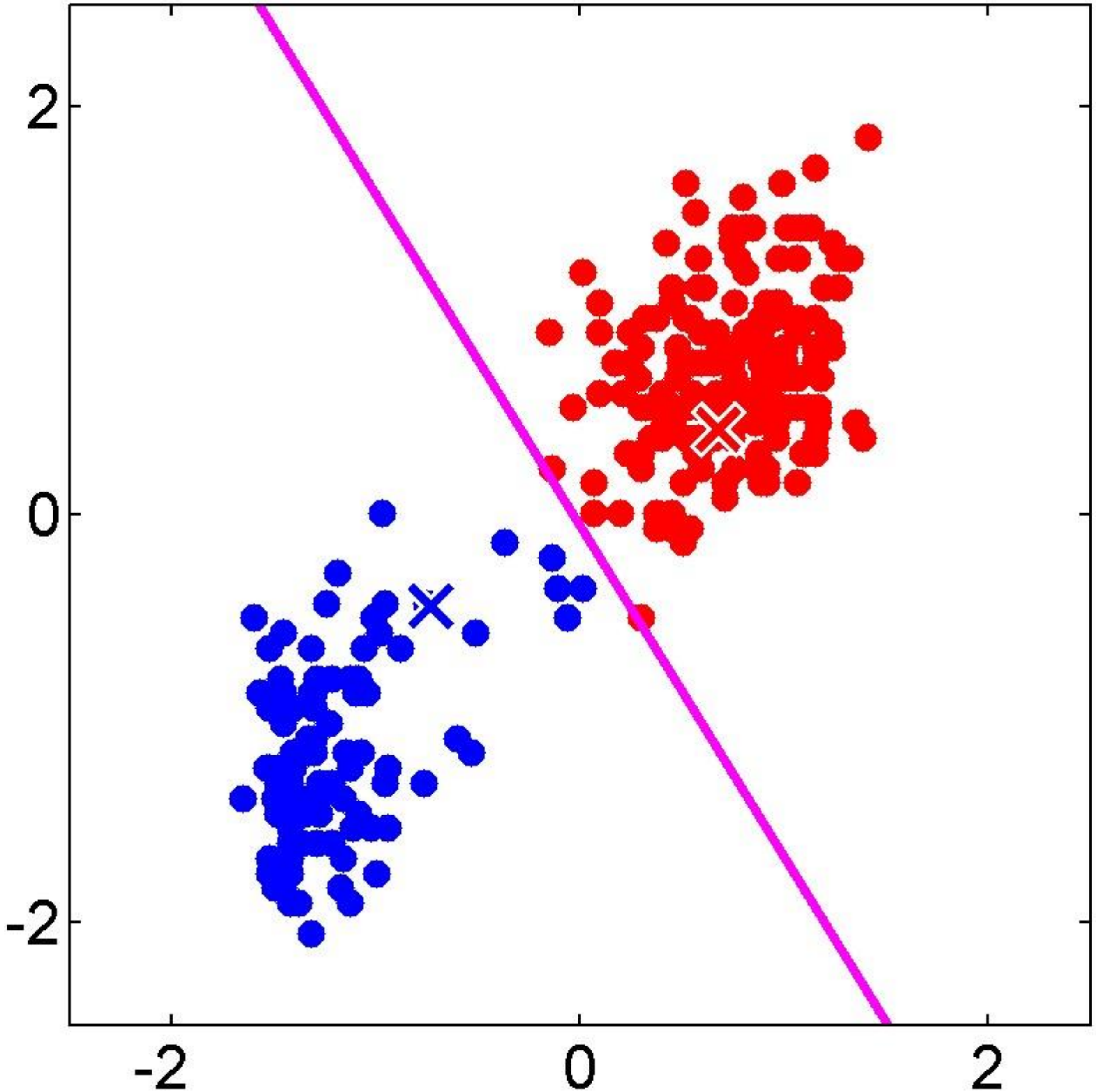
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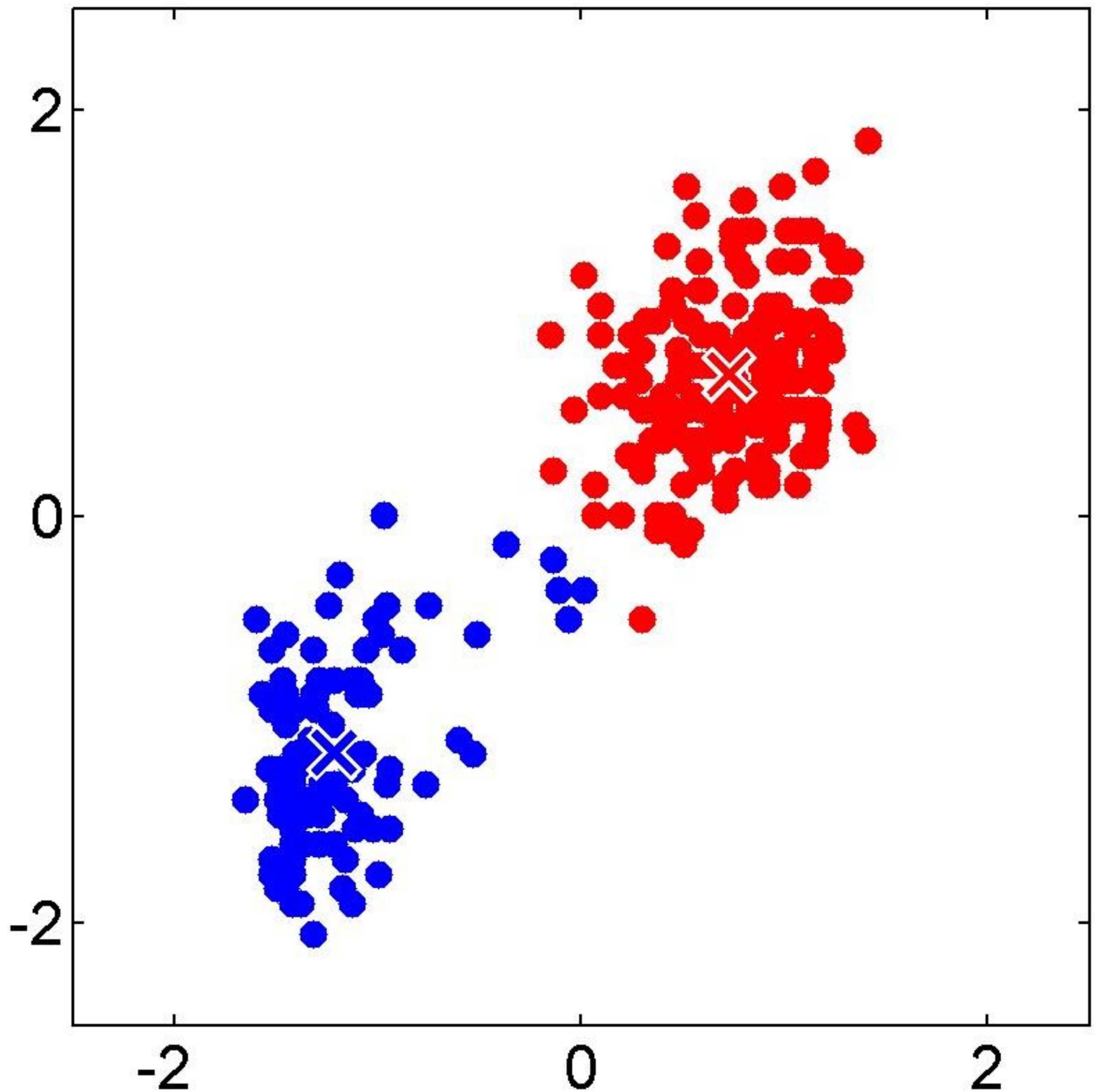
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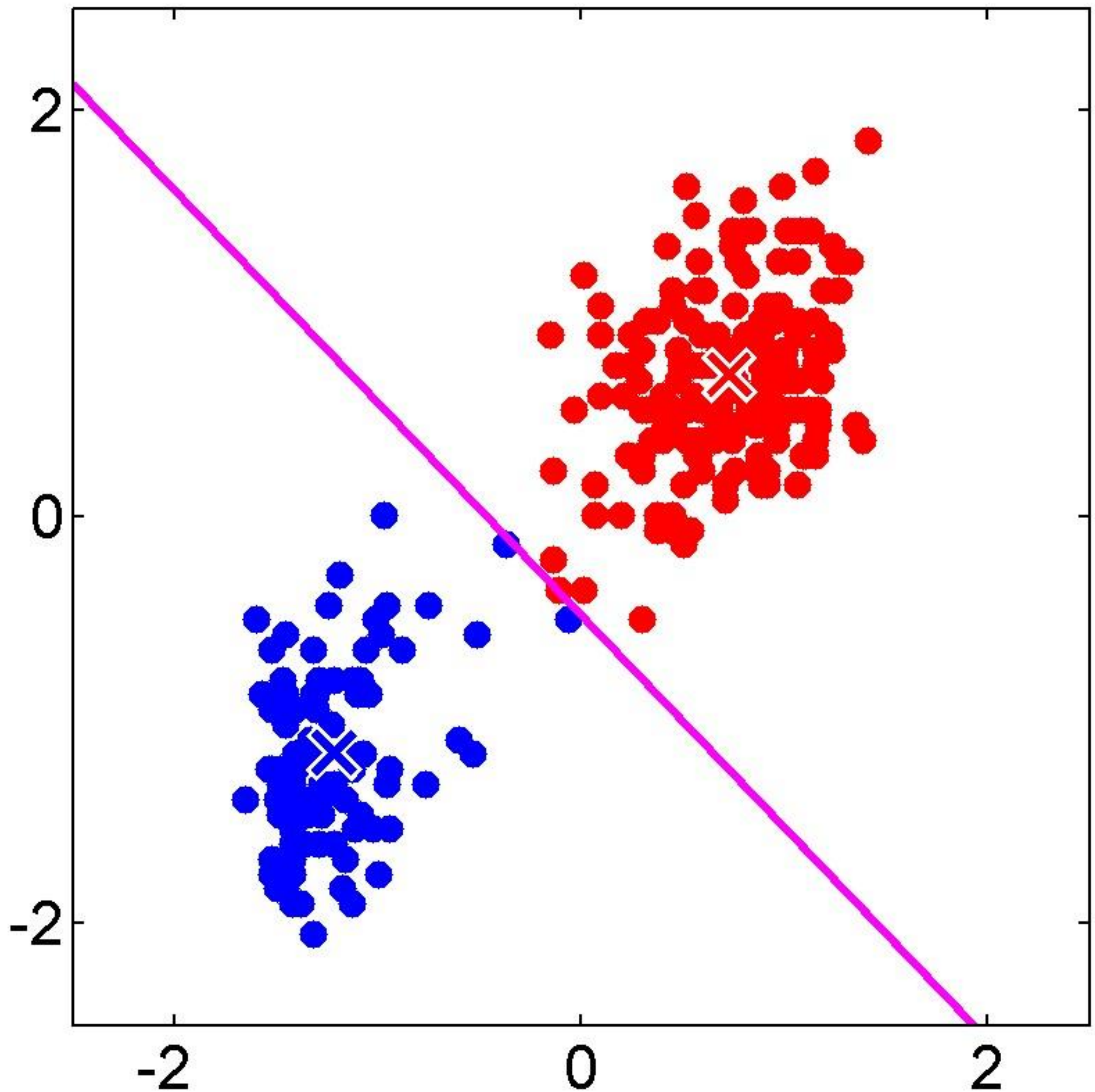

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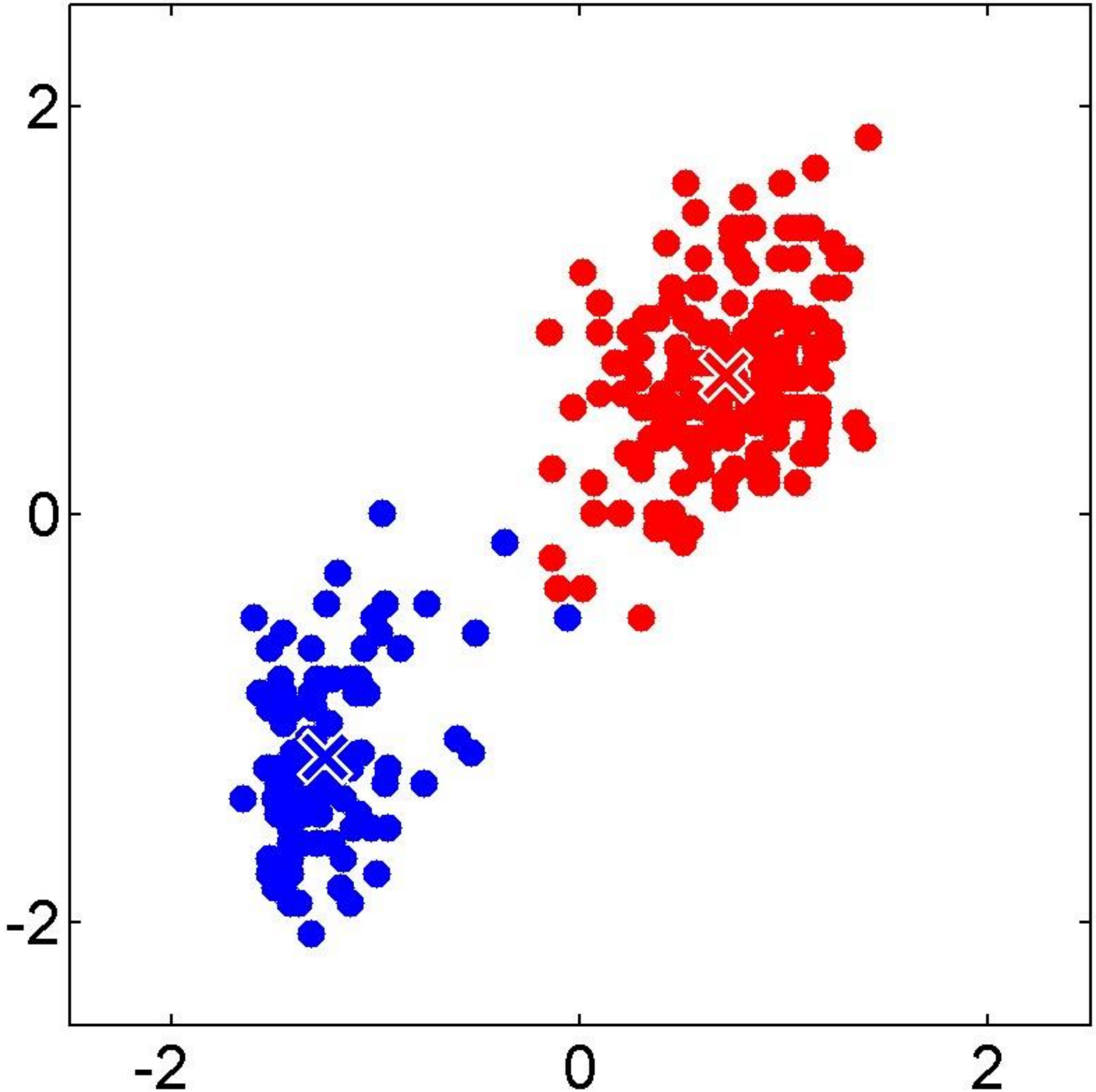
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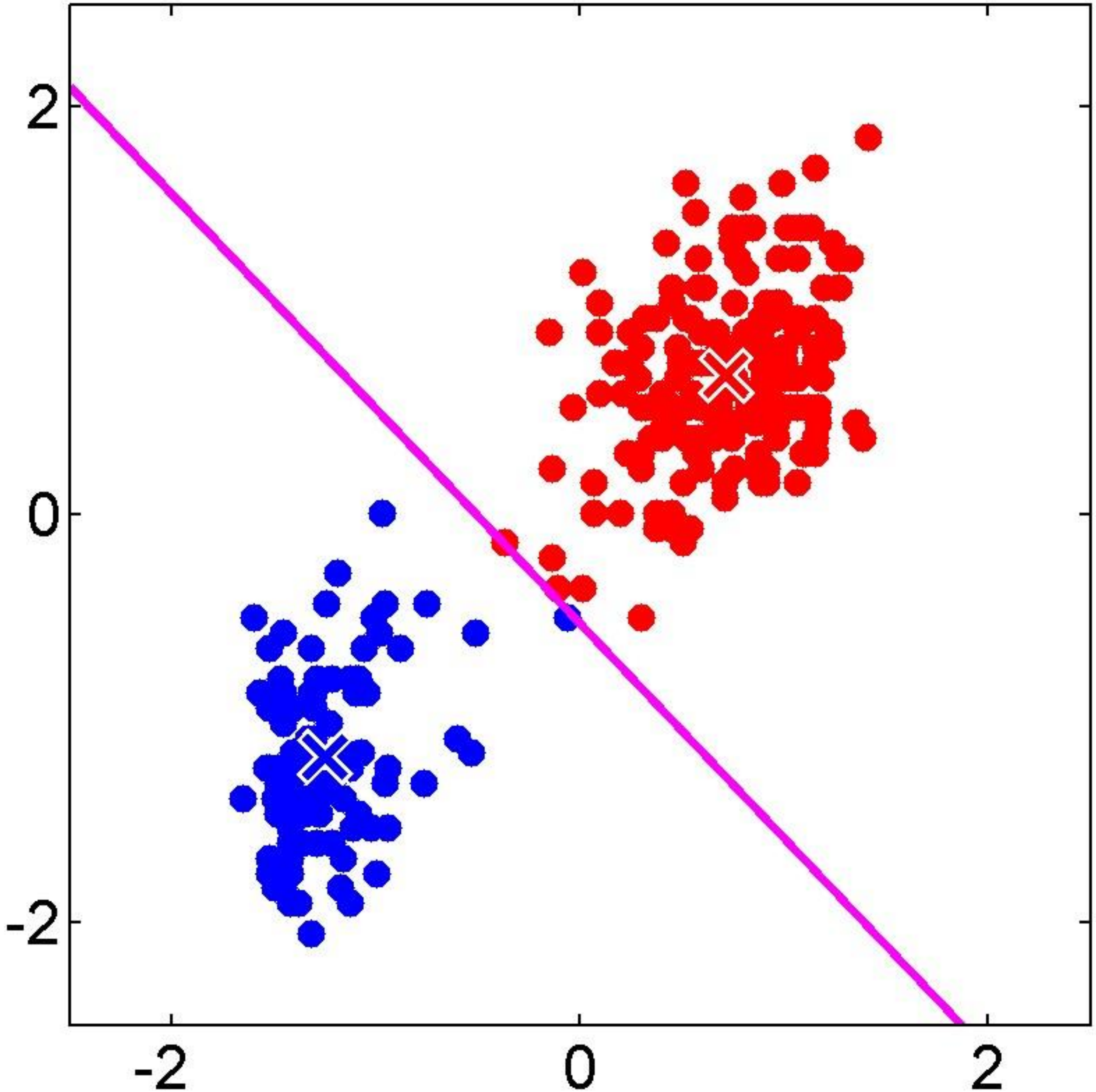
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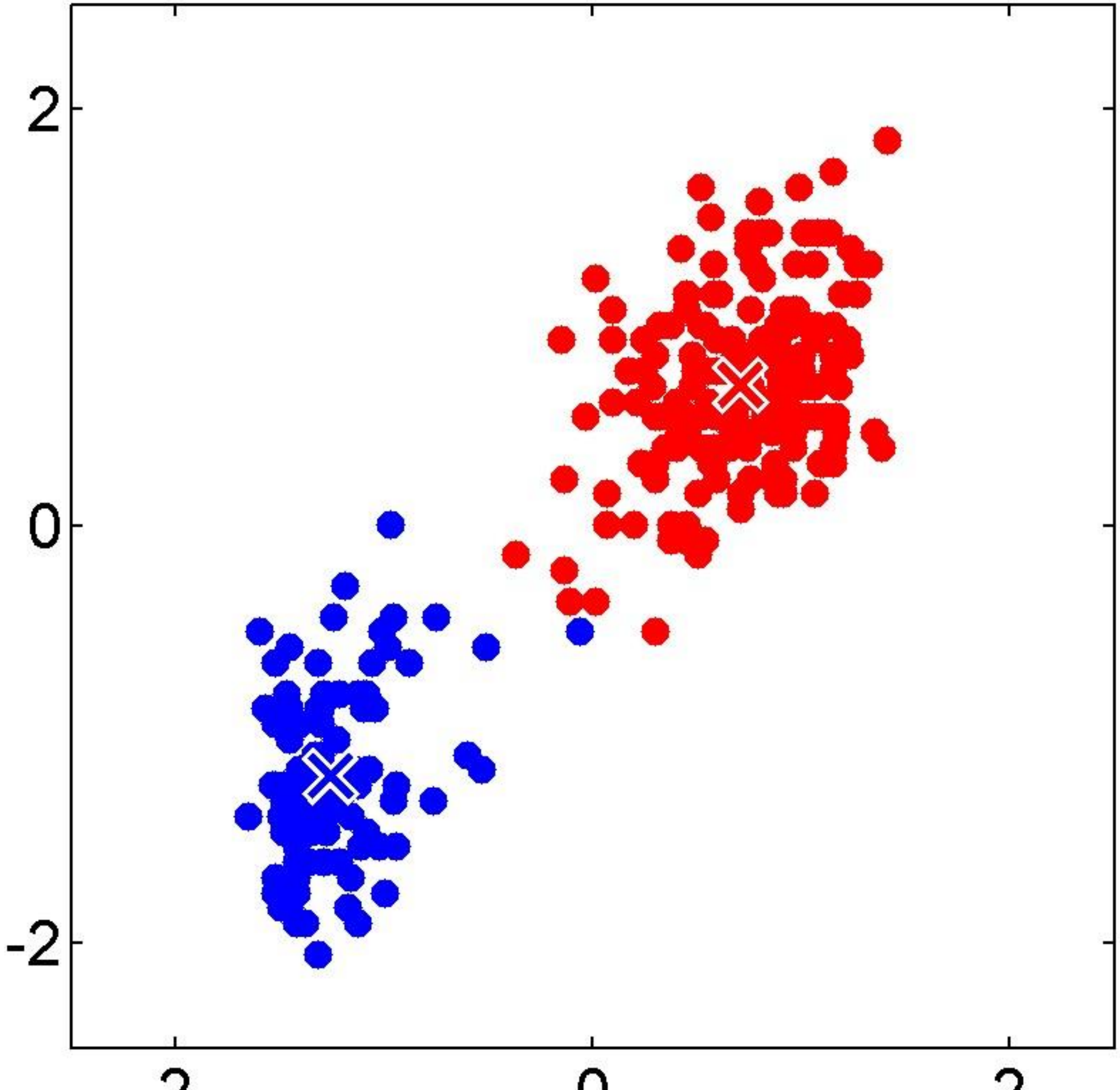
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