Digital Signal Processing

Chap 7. Filter Design Techniques

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Filters

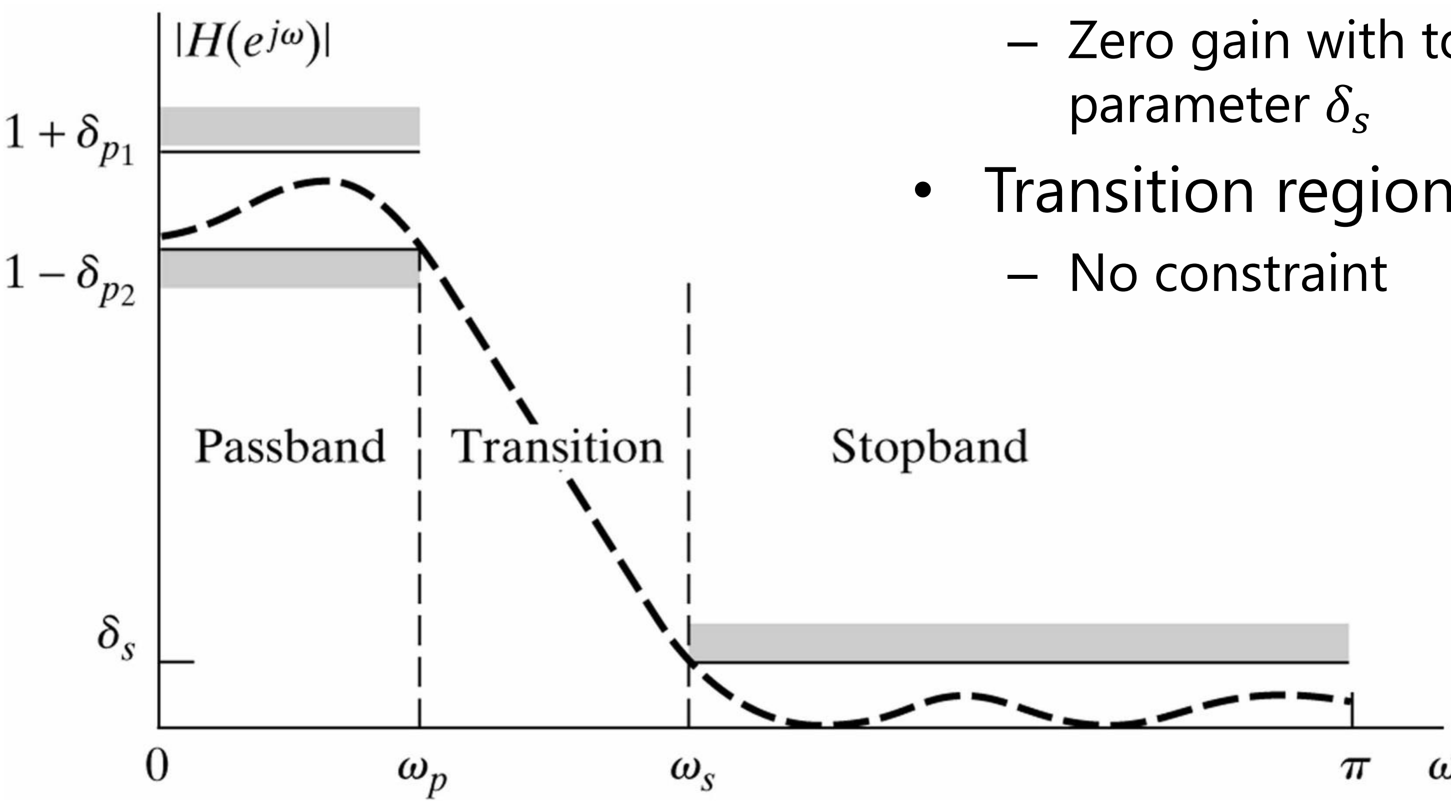
- **Allow some parts** of input materials to pass through it
- but **block the other parts**

Summary

- Filter specifications
- Design of DT IIR filters from CT filters
  - Impulse invariance
  - Bilinear transformation
  - Ex) Butterworth, Chebyshev, Elliptic filters
- There are many more filters and their design techniques, which are beyond the scope of this course.

Filter Specifications

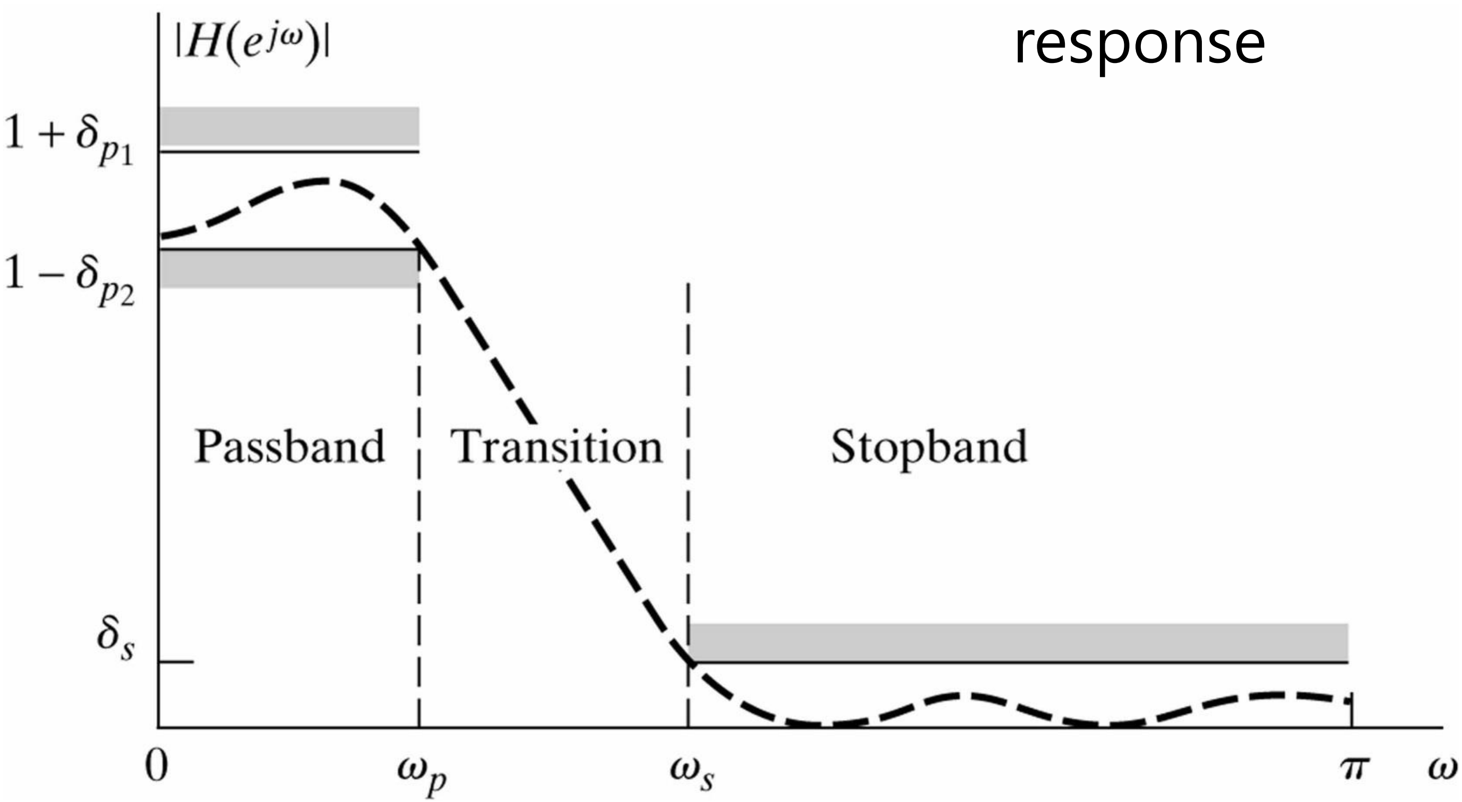
Tolerance Scheme



- Passband
  - Unity gain with tolerance parameters δ_{p_1} and δ_{p_2}
- Stopband
  - Zero gain with tolerance parameter δ_s
- Transition region
  - No constraint

Filter Specifications

Tolerance Scheme



- We focus on the case of lowpass filter with no constraint on the phase response

Design of DT filters from CT filters

- Historically, CT filters had been more intensively researched than DT filters
- It was natural to consider converting CT filters into DT filters
- Two such methods are
  - Impulse invariance
  - Bilinear transformation

Impulse Invariance

- Recall the C/D conversion
  - $x[n] = x_c(nT)$
  - $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$
- Similarly, we obtain a DT filter from a CT filter by

$$h[n] = T_d h_c(nT_d)$$

Then

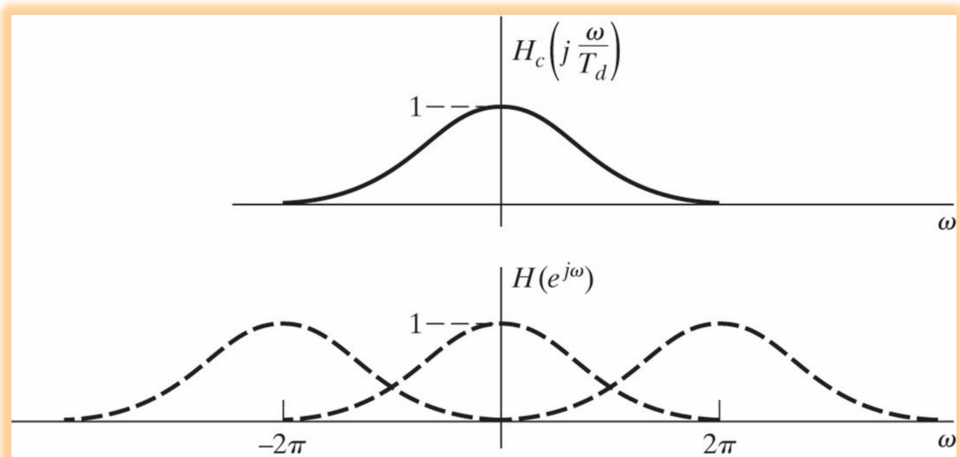
$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T_d} - \frac{2\pi k}{T_d}\right)\right)$$

Impulse Invariance

- If the CT filter is band-limited, *i.e.* $H_c(j\Omega) = 0$ for $|\Omega| > \frac{\pi}{T_d}$,

$$H(e^{j\omega}) = H_c\left(j\frac{\omega}{T_d}\right), \qquad |\omega| \leq \pi$$

- In practice, aliasing occurs and the effect should be checked after the design
- The design sampling period T_d can be set to any value
  - It does not affect the implemented filter



Impulse Invariance

- Step 1: Transform the DT filter specifications to CT filter specifications
$$H_c(j\Omega) = H(e^{j\Omega T_d})$$
- Step 2: Obtain a CT filter $H_c(s)$
- Step 3: $H_c(s) \Rightarrow h_c(t) \Rightarrow h[n] \Rightarrow H(z)$
  - ex) $H_c(s) = \frac{A_k}{s-s_k} \Rightarrow H(z) = \frac{T_d A_k}{1-e^{s_k T_d} z^{-1}}$
- Step 4: Check the aliasing effect
  - If the DT specifications are not satisfied, restart with stronger constraints.

Ex) Butterworth Filter Design Using Impulse Invariance

- Step 1: Given DT filter specifications

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1, \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi$$

we have CT specifications with $T_d = 1$

$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 0.2\pi$$

$$|H_c(j\Omega)| \le 0.17783, \qquad 0.3\pi \le |\Omega| \le \pi$$

Ex) Butterworth Filter Design Using Impulse Invariance

- Step 2: Butterworth filter

$$|H_c(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$1+\left(\frac{0.2\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \text{ and } 1+\left(\frac{0.3\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

$\Rightarrow N = 5.8858 \simeq 6$ and $\Omega_c = 0.70474$

Then, based on the CT signal processing techniques

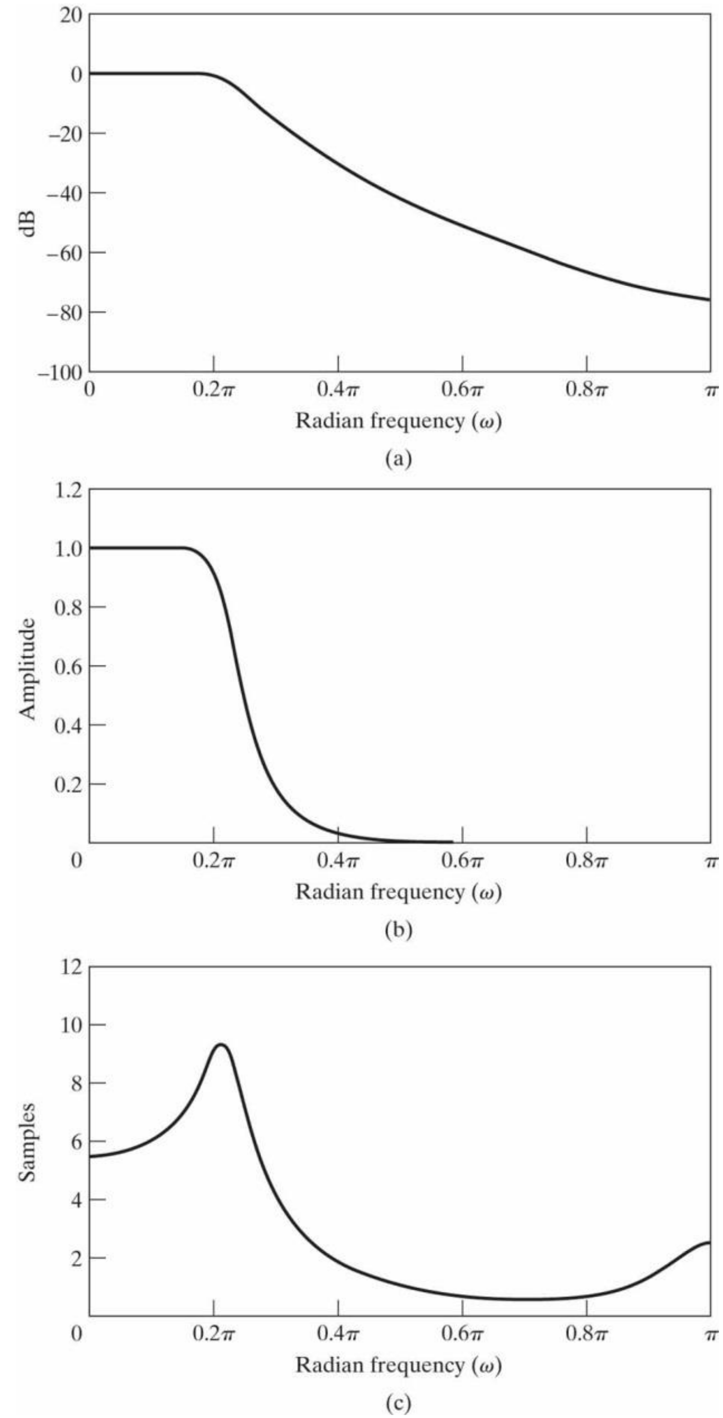
$$H_c(s) = \frac{0.12}{(s^2+0.36s+0.49)(s^2+0.99s+0.49)(s^2+1.36s+0.49)}$$

- Step 3: Digital Butterworth filter

$$H(z) = \frac{0.29-0.45z^{-1}}{1-1.30z^{-1}+0.69z^{-2}} + \frac{-2.14+1.15z^{-1}}{1-1.07z^{-1}+0.37z^{-2}} + \frac{1.86-0.63z^{-1}}{1-1.00z^{-1}+0.26z^{-2}}$$

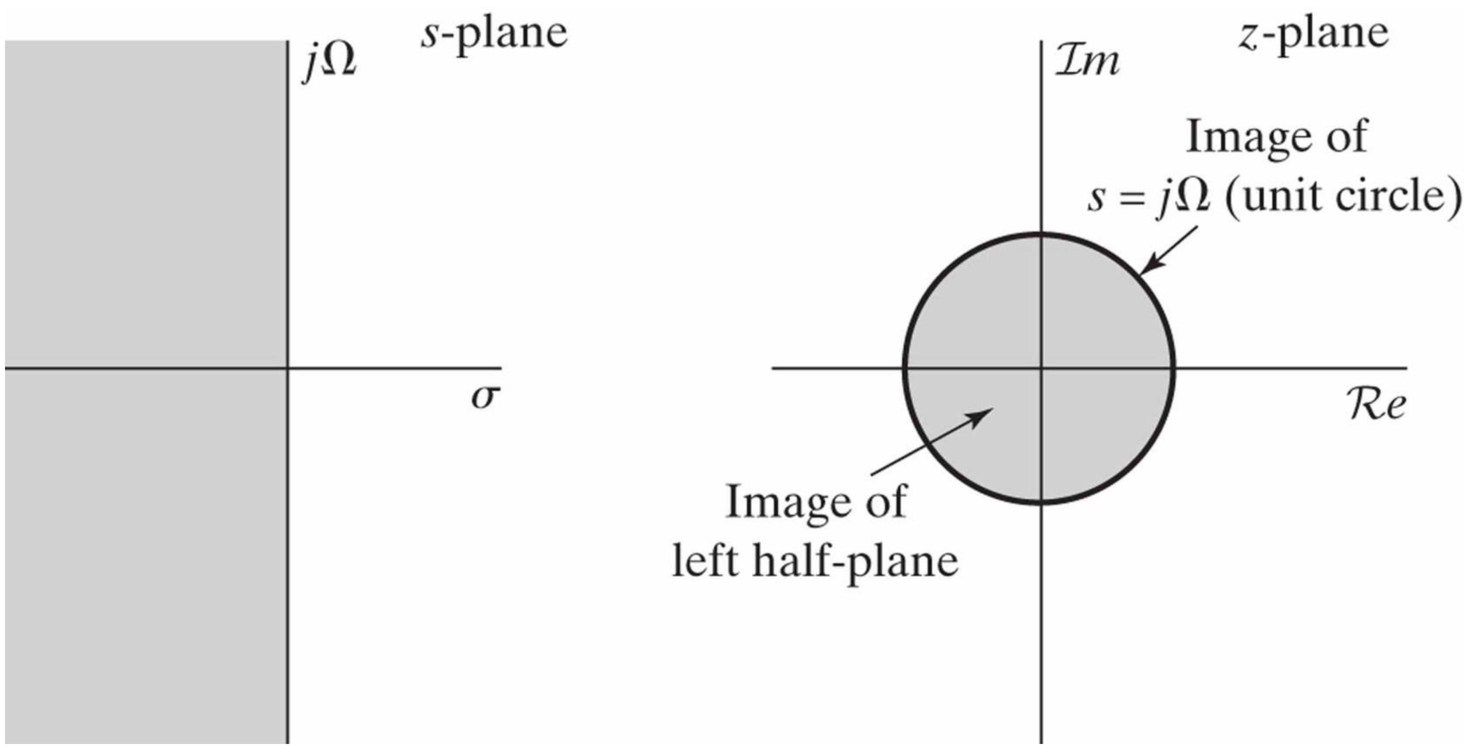
Ex) Butterworth Filter Design Using Impulse Invariance

- Step 4: Check



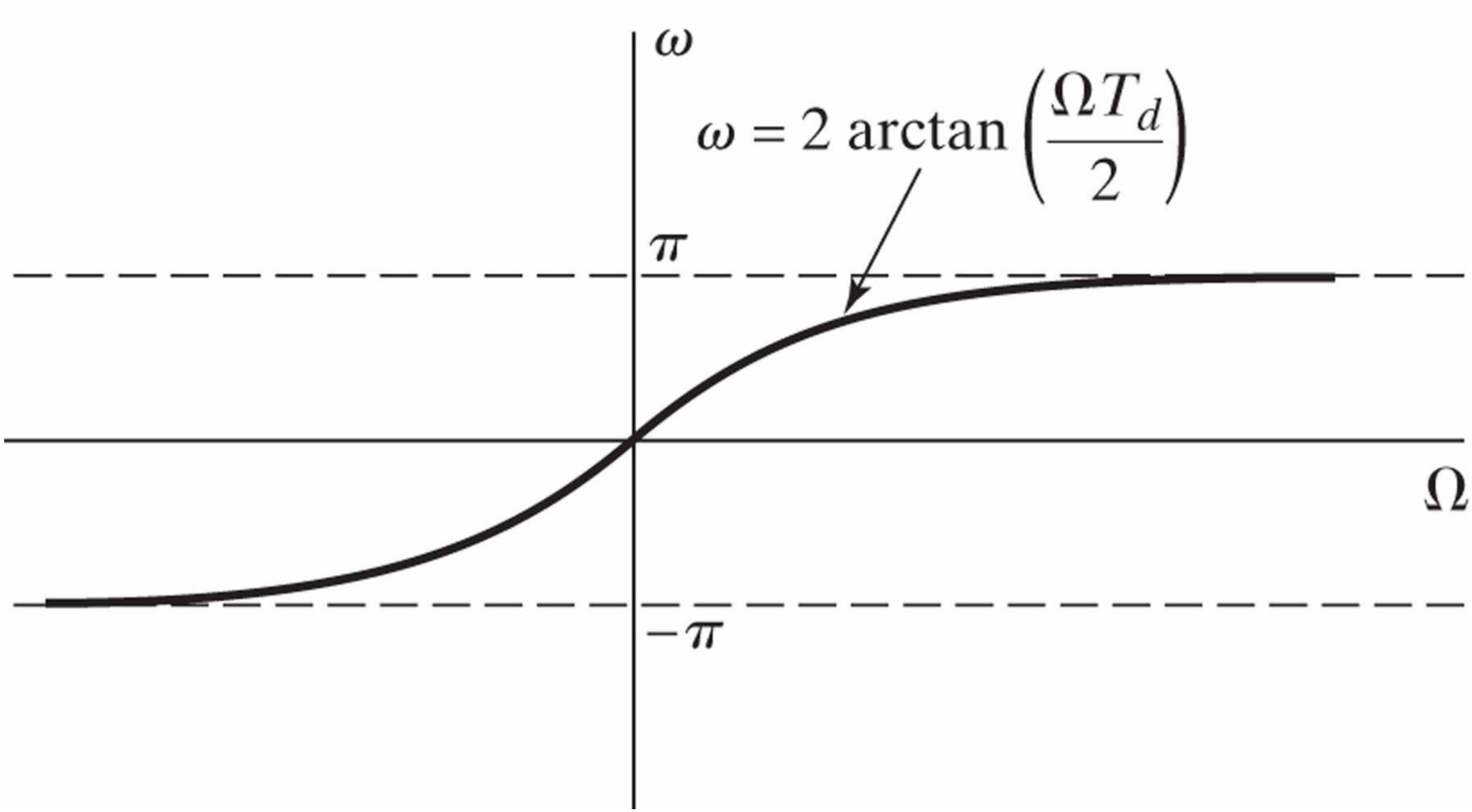
Bilinear Transformation

- $s = \frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ or $z = \frac{1+\frac{T_d}{2}s}{1-\frac{T_d}{2}s}$

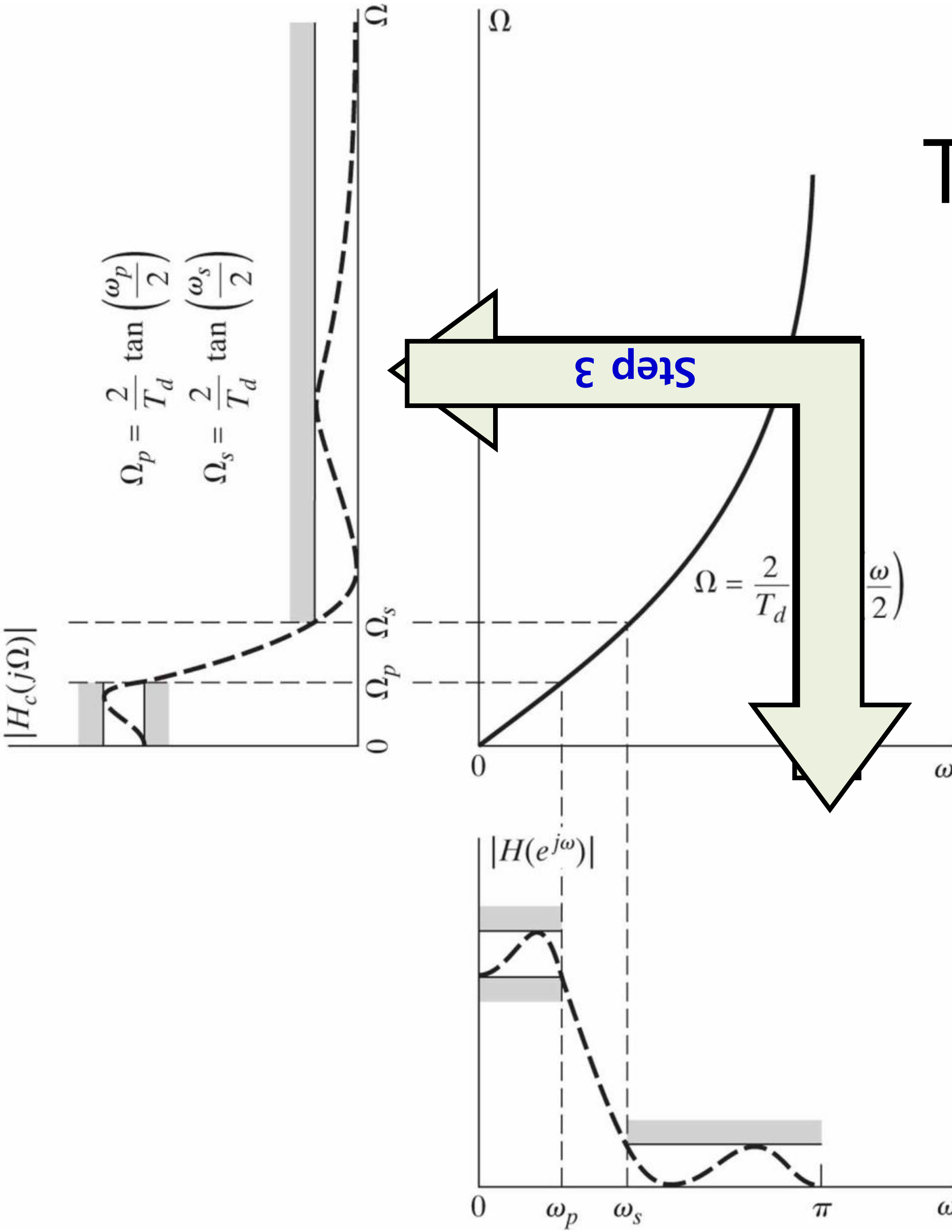


Bilinear Transformation

- $\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)$ or $\omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$



Bilinear Transformation



- Step 1: Transform the DT filter spec. to CT filter spec.
- Step 2: Obtain a CT filter $H_c(s)$
- Step 3: $H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$

Ex) Butterworth Filter Design Using Bilinear Transformation

- Step 1: Given DT filter specifications

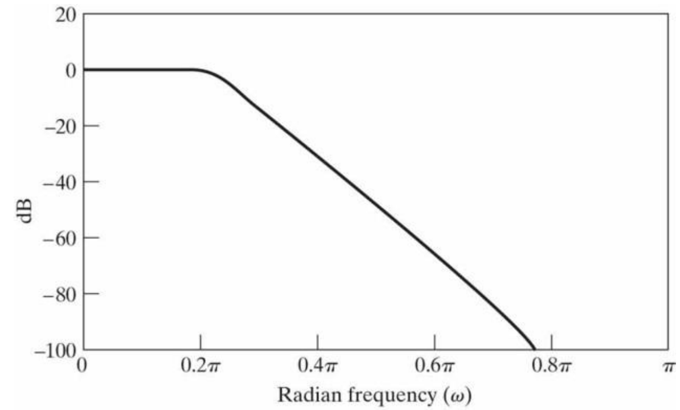
$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1, \qquad 0 \le |\omega| \le 0.2\pi$$
$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi$$

we have CT specifications with $T_d = 1$

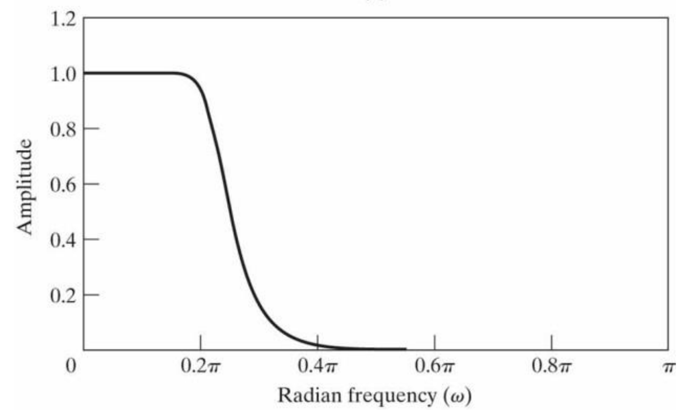
$$0.89125 \le |H_c(j\Omega)| \le 1, \qquad 0 \le |\Omega| \le 2\tan\left(\frac{0.2\pi}{2}\right)$$
$$|H_c(j\Omega)| \le 0.17783, \qquad 2\tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| \le \infty$$

- Step 2: Obtain a CT Butterworth filter
- Step 3: $H(z) = H_c\left(2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$

Ex) Butterworth Filter Design Using Bilinear Transformation



(a)



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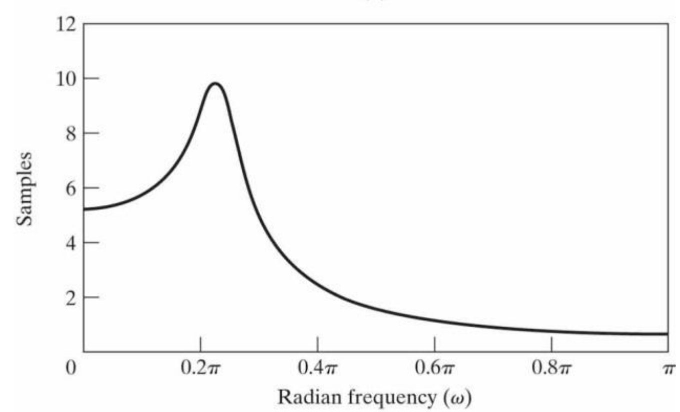


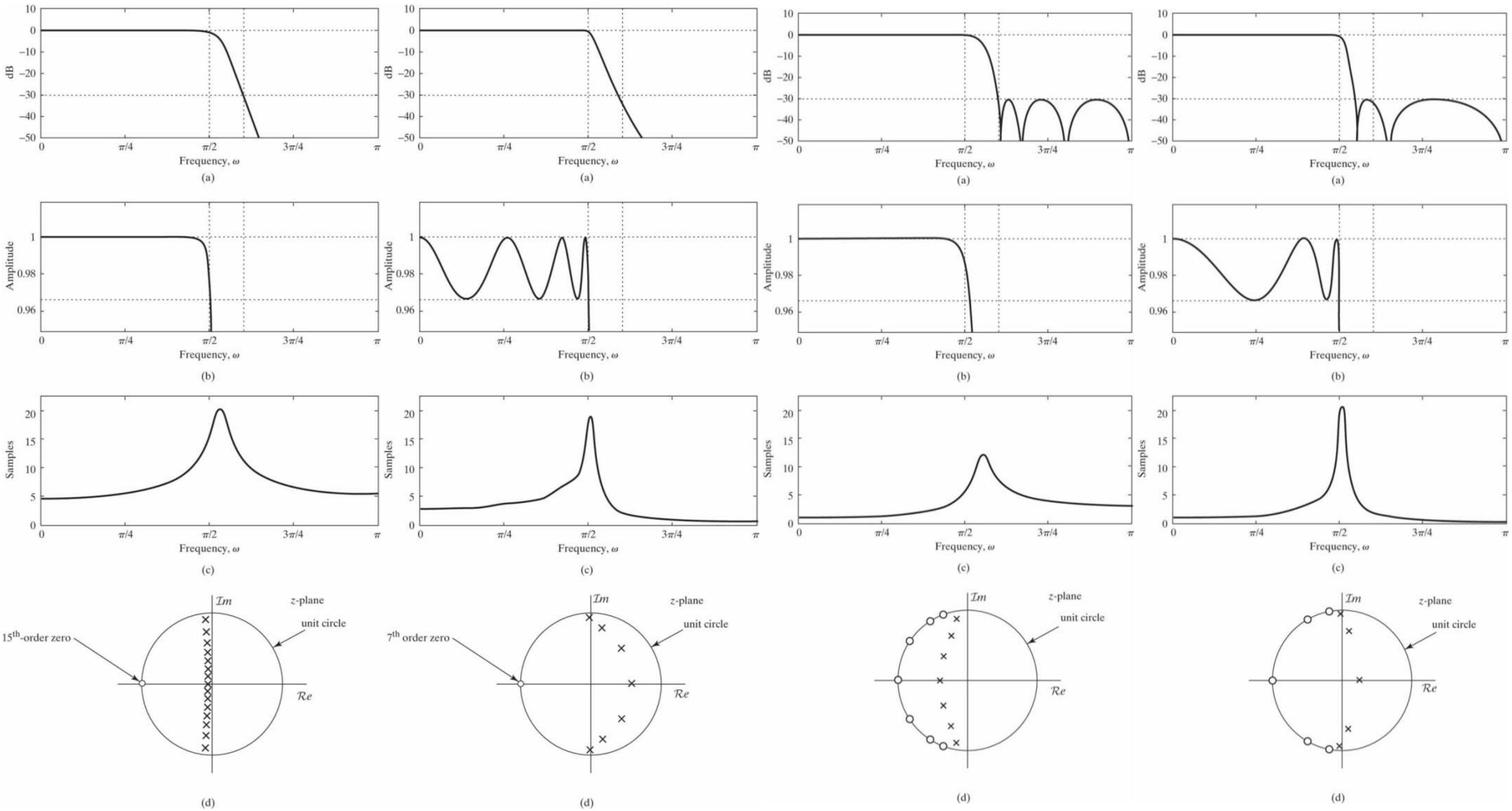
(c)

Comparison of Butterworth, Chebyshev, and Elliptic Filters

- Specifications
  - Passband edge frequency $\omega_p = 0.5\pi$
  - Stopband edge frequency $\omega_s = 0.6\pi$
  - Maximum passband gain = 0 dB
  - Minimum passband gain = -0.3dB
  - Maximum stopband gain = -30dB

Comparison of Butterworth, Chebyshev, and Elliptic Filters

Monotone | Ripple in passband | Ripple in stopband | Ripples!



Butterworth, 15th | **Chebyshev I, 7th** | **Chebyshev II, 7th** | **Elliptic, 5th**