Chapter 9

Sum of Random Variables

Chang-Su Kim

Korea University

Section 9.1

Expected Values of Sums

Theorem 9.1

For any set of random variables $X_1, \ldots, X_n$, the sum $W_n = X_1 + \cdots + X_n$ has expected value

$$\mathsf{E}[W_n] = \mathsf{E}[X_1] + \mathsf{E}[X_2] + \cdots + \mathsf{E}[X_n].$$

Theorem 9.2

The variance of $W_n = X_1 + \cdots + X_n$ is

$$\mathrm{Var}[W_n] = \sum_{i=1}^{n} \mathrm{Var}[X_i] + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{Cov}\left[X_i, X_j\right].$$

Theorem 9.3

When $X_1, \ldots, X_n$ are uncorrelated,

$$\mathsf{Var}[W_n] = \mathsf{Var}[X_1] + \cdots + \mathsf{Var}[X_n].$$

Example 9.1 Problem

$X_0, X_1, X_2, \ldots$ is a sequence of random variables with expected values $\mathrm{E}[X_i] = 0$ and covariances, $\mathrm{Cov}[X_i, X_j] = 0.8^{|i-j|}$. Find the expected value and variance of a random variable Y_i defined as the sum of three consecutive values of the random sequence

$$Y_i = X_i + X_{i-1} + X_{i-2}. \tag{9.3}$$

Quiz 9.1

Let W_n denote the sum of n independent throws of a fair four-sided die. Find the expected value and variance of W_n.

Section 9.2

Moment Generating Functions

Definition 9.1 (MGF)

For a random variable X, the moment generating function (MGF) *of X is*

$$\phi_X(s) = \mathsf{E}\left[e^{sX}\right].$$

Theorem 9.4

A random variable X with MGF $\phi_X(s)$ has nth moment

$$\mathsf{E}[X^n] = \left.\frac{d^n \phi_X(s)}{ds^n}\right|_{s=0}.$$

Random Variable	**PMF or PDF**	**MGF $\Phi_X(s)$**
Bernoulli (p)	$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$	$1-p+pe^s$
Binomial (n,p)	$P_X(x) = \binom{n}{x} p^x (1-p)^x$	$(1-p+pe^s)^n$
Poisson (α)	$P_X(x) = \frac{\alpha^x e^{-\alpha}}{x!}$	$e^{\alpha(e^s-1)}$
Exponential (λ)	$f_X(x) = \lambda e^{-\lambda x}$	$\frac{\lambda}{\lambda - s}$
Erlang (n,λ)	$f_X(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!}$	$\left(\frac{\lambda}{\lambda - s}\right)^n$
Gaussian (μ,σ)	$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{s\mu + \frac{s^2\sigma^2}{2}}$

Example 9.4 Problem

X is an exponential random variable with MGF $\phi_X(s) = \lambda/(\lambda - s)$. What are the first and second moments of X? Write a general expression for the nth moment.

Quiz 9.2

Random variable K has PMF

$$P_K(k) = \begin{cases} 0.2 & k = 0, \ldots, 4, \\ 0 & \text{otherwise.} \end{cases} \tag{9.26}$$

Use $\phi_K(s)$ to find the first, second, third, and fourth moments of K.

Section 9.3

MGF of the Sum of Independent Random Variables

Theorem 9.6

For a set of independent random variables $X_1, \ldots, X_n$, the moment generating function of $W = X_1 + \cdots + X_n$ is

$$\phi_W(s) = \phi_{X_1}(s)\phi_{X_2}(s)\cdots\phi_{X_n}(s).$$

When $X_1, \ldots, X_n$ are iid, each with MGF $\phi_{X_i}(s) = \phi_X(s)$,

$$\phi_W(s) = [\phi_X(s)]^n\,.$$

Example 9.5 Problem

J and K are independent random variables with probability mass functions

$$\begin{array}{c|ccc} j & 1 & 2 & 3 \\ \hline P_J(j) & 0.2 & 0.6 & 0.2 \end{array}, \qquad \begin{array}{c|cc} k & -1 & 1 \\ \hline P_K(k) & 0.5 & 0.5 \end{array}. \tag{9.31}$$

Find the MGF of $M = J + K$. What are $P_M(m)$ and $E[M^3]$?

Theorem 9.7

If $K_1, \ldots, K_n$ are independent Poisson random variables, $W = K_1 + \cdots + K_n$ is a Poisson random variable.

Theorem 9.8

The sum of n independent Gaussian random variables $W = X_1 + \cdots + X_n$ is a Gaussian random variable.

Quiz 9.3(A)

Let $K_1, K_2, \ldots, K_m$ be iid discrete uniform random variables with PMF

$$P_K(k) = \begin{cases} 1/n & k = 1, 2, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \tag{9.40}$$

Find the MGF of $J = K_1 + \cdots + K_m$.

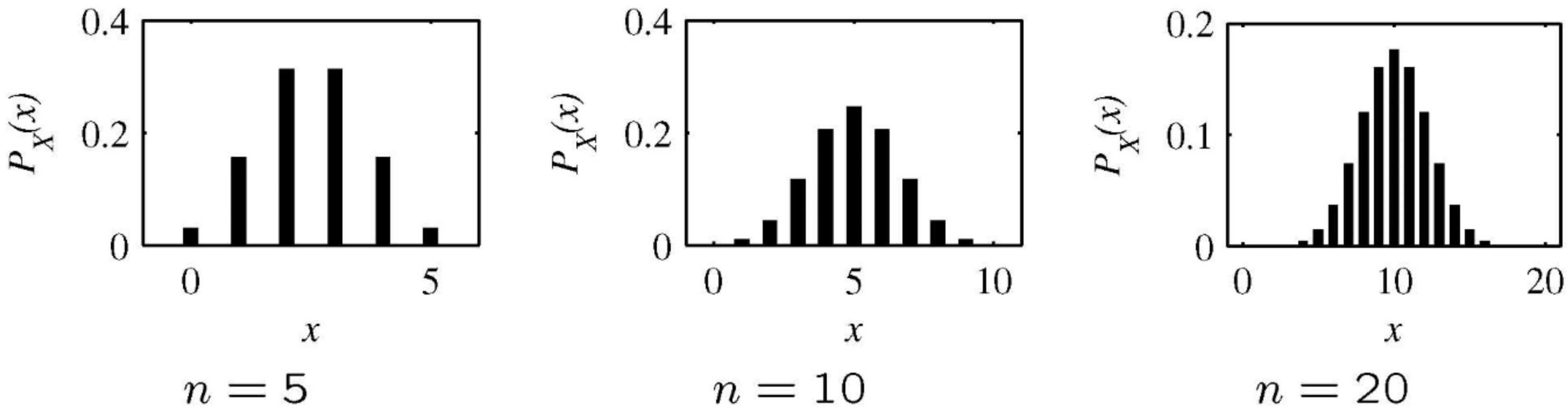
Quiz 9.3(B)

Let $X_1, \ldots, X_n$ be independent Gaussian random variables with $\mathrm{E}[X_i = 0]$ and $\mathrm{Var}[X_i] = i$. Find the PDF of

$$W = \alpha X_1 + \alpha^2 X_2 + \cdots + \alpha^n X_n. \tag{9.41}$$

Central Limit Theorem

Figure 9.1



The PMF of the X, the number of heads in n coin flips for $n = 5, 10, 20$. As n increases, the PMF more closely resembles a bell-shaped curve.

Theorem 9.10 Central Limit Theorem

Given $X_1, X_2, \ldots$, a sequence of iid random variables with expected value μ_X and variance σ_X^2, the CDF of $Z_n = (\sum_{i=1}^n X_i - n\mu_X)/\sqrt{n\sigma_X^2}$ has the property

$$\lim_{n\to\infty} F_{Z_n}(z) = \Phi(z).$$

Definition 9.2 Approximation

Let $W_n = X_1 + \cdots + X_n$ *be the sum of* n *iid random variables, each with* $\mathsf{E}[X] = \mu_X$ *and* $\mathsf{Var}[X] = \sigma_X^2$*. The central limit theorem approximation to the CDF of* W_n *is*

$$F_{W_n}(w) \approx \Phi\left(\frac{w - n\mu_X}{\sqrt{n\sigma_X^2}}\right).$$

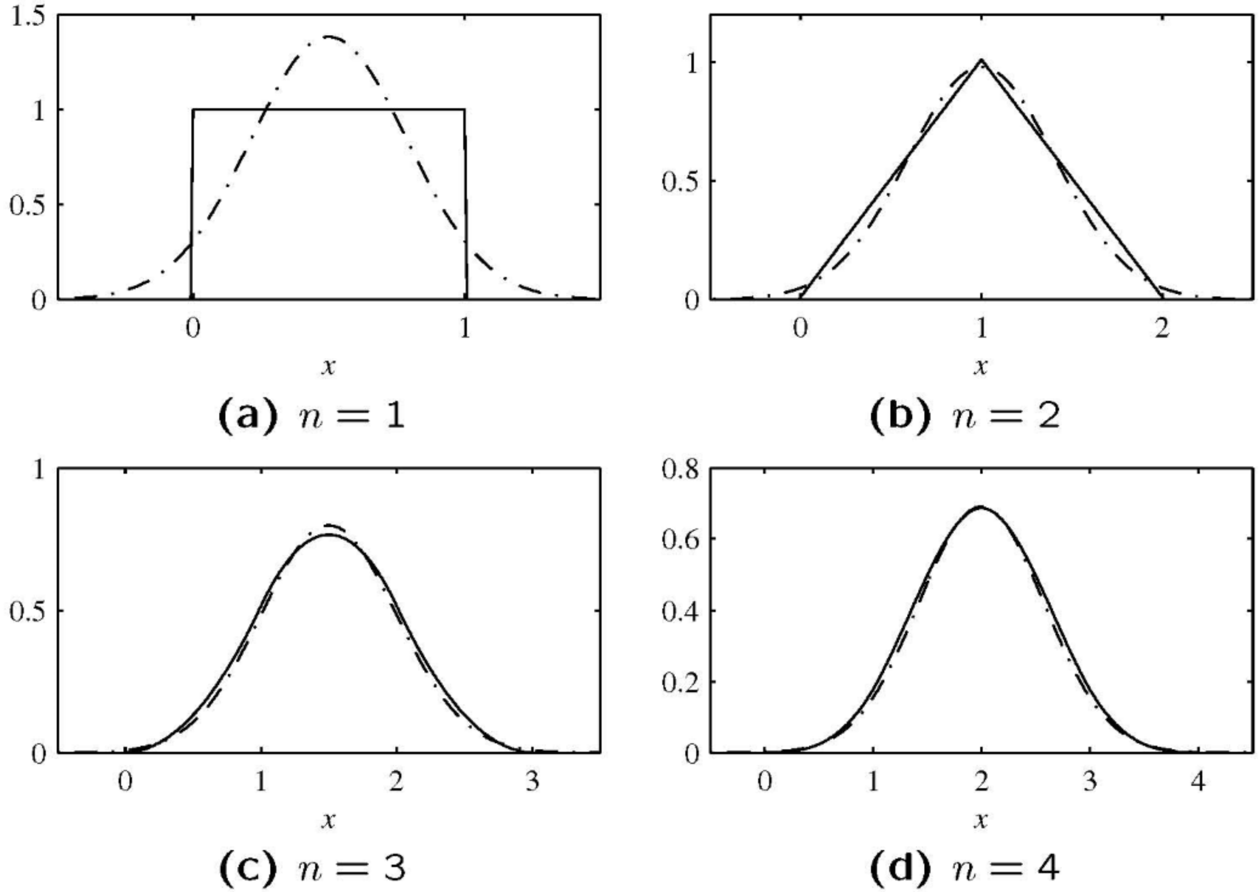
Example 9.6

To gain some intuition into the central limit theorem, consider a sequence of iid continuous random variables X_i, where each random variable is uniform (0,1). Let

$$W_n = X_1 + \cdots + X_n. \tag{9.46}$$

Recall that $\mathrm{E}[X] = 0.5$ and $\mathrm{Var}[X] = 1/12$. Therefore, W_n has expected value $\mathrm{E}[W_n] = n/2$ and variance $n/12$. The central limit theorem says that *the CDF* of W_n should approach a Gaussian CDF with the same expected value and variance. Moreover, since W_n is a continuous random variable, we would also expect that the PDF of W_n would converge to a Gaussian PDF. In Figure 9.2, we compare the PDF of W_n to the PDF of a Gaussian random variable with the same expected value and variance. First, W_1 is a uniform random variable with the rectangular PDF shown in Figure 9.2(a). This figure also shows the PDF of W_1, a Gaussian random variable with expected value $\mu = 0.5$ and variance $\sigma^2 = 1/12$. Here the PDFs are very dissimilar. When we consider $n = 2$, we have the situation in Figure 9.2(b). The PDF of W_2 is a triangle with expected value 1 and variance 2/12. The figure shows the corresponding Gaussian PDF. The following figures show the PDFs of $W_3, \ldots, W_6$. The convergence to a bell shape is apparent.

Figure 9.2



The PDF of W_n, the sum of n uniform $(0, 1)$ random variables, and the corresponding central limit theorem approximation for $n = 1, 2, 3, 4$. The solid — line denotes the PDF $f_{W_n}(w)$, and the broken $- \cdot -$ line denotes the Gaussian approximation.

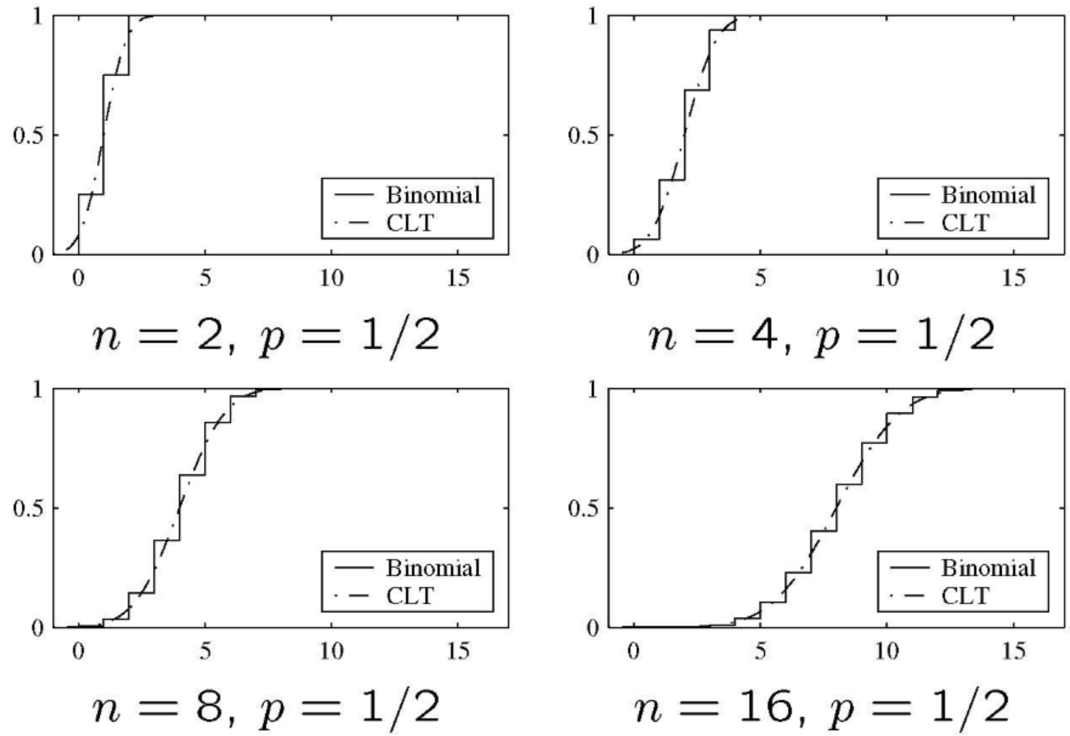
Example 9.7

Now suppose $W_n = X_1 + \cdots + X_n$ is a sum of independent Bernoulli (p) random variables. We know that W_n has the binomial PMF

$$P_{W_n}(w) = \binom{n}{w} p^w (1-p)^{n-w}. \tag{9.47}$$

No matter how large n becomes, W_n is always a discrete random variable and would have a PDF consisting of impulses. However, the central limit theorem says that the CDF of W_n converges to a Gaussian CDF. Figure 9.3 demonstrates the convergence of the sequence of binomial CDFs to a Gaussian CDF for $p = 1/2$ and four values of n, the number of Bernoulli random variables that are added to produce a binomial random variable. For $n \geq 32$, Figure 9.3 suggests that approximations based on the Gaussian distribution are very accurate.

Figure 9.3



The binomial (n, p) CDF and the corresponding central limit theorem approximation for $n = 4, 8, 16, 32$, and $p = 1/2$.

Example 9.9 Problem

Transmit one million bits. Let A denote the event that there are at least 499,000 ones but no more than 501,000 ones. What is $\mathrm{P}[A]$?

Quiz 9.4

X milliseconds, the total access time (waiting time + read time) to get one block of information from a computer disk, is the continuous (0,12) random variable. Before performing a certain task, the computer must access 12 different blocks of information from the disk. (Access times for different blocks are independent of one another.) The total access time for all the information is a random variable A milliseconds.

(a) Find the expected value and variance of the access time X.

(b) Find the expected value and standard deviation of the total access time A.

(c) Use the central limit theorem to estimate P[$A > 75$ ms].

(d) Use the central limit theorem to estimate P[$A < 48$ ms].